Differences From Special Relativity

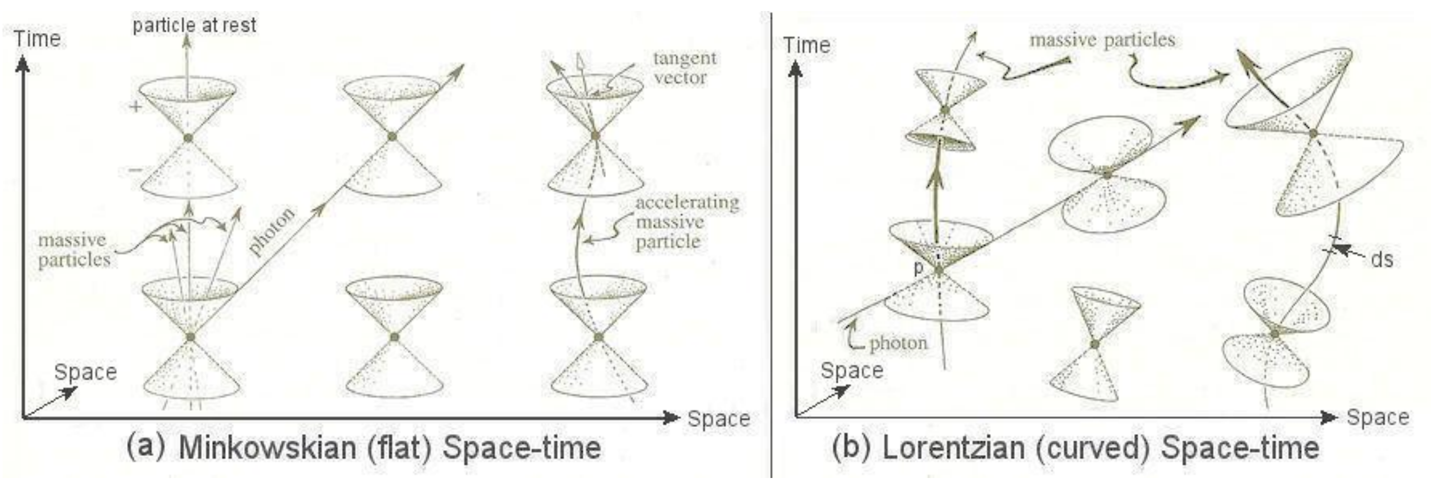
Conservation Laws

Some of the tensors we have discussed are mass and charge, both rank-0 tensors, and the rank-1 momentum tensor (which contains both classical energy and momentum), all of which are conserved. This makes it natural to ask how conservation laws are formulated in general relativity. For example, in classical physics the total amount of mass in the universe stays constant. Relativity allows us to make physical models of the universe as a whole, so it seems as though we ought to be able to talk about conservation laws in relativity. But we can't– time is relative, so we can't talk about the amount of something "before" and the amount "after". And for vectors, we can't even compare the momentum (for example) at one event to the momentum at another without using parallel transport, which is path-dependent. Finally, the energy of matter is not conserved in the presence of changing spacetime curvature, but changes in response to it. The conclusion is that **there are no global principles of conservation in general relativity**.

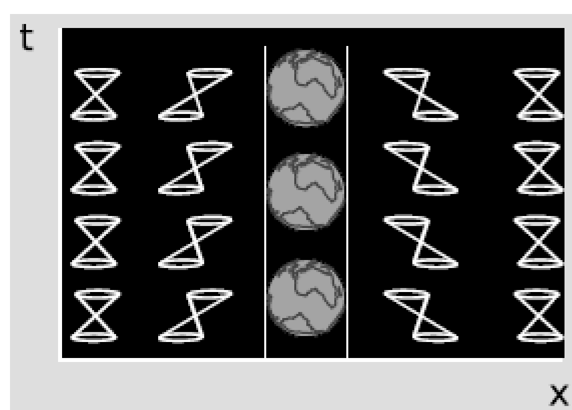
But under certain circumstances we can deduce that some quantities (like energy and angular momentum) are locally conserved (usually due to symmetries), such as for the Schwarzschild or Kerr solutions. In addition, any time the spacetime gets close to flat far away from the system we're studying (*asymptotically flat*), then we can define a far-away boundary that surrounds the system (a global Lorentz frame) as a reference, and parallel-transport loses its path-dependence, which allows us to deduce some conservation rules. See the "Stress-Energy-Momentum Tensor" and "Energy Conditions" sections below for more on conservation.

Light Cones

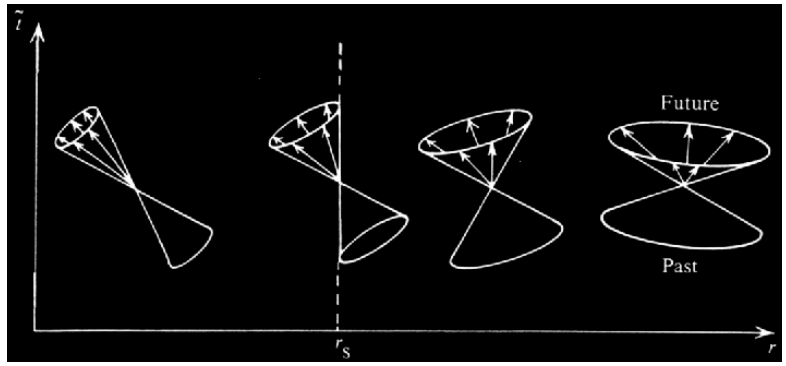
In the uniform Minkowski spacetime of special relativity, all light cones line up with their axes parallel with one another. However, this is no longer necessarily true in the curved spacetime of general relativity :



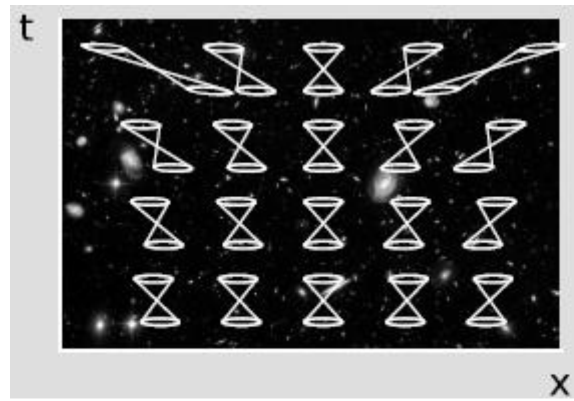
And when gravity sources are present, **light cones near gravitational sources tip toward the sources** :



In the case of a black hole, the light cone tips over so far that eventually the entire future timelike region lies within the black hole (at r = r_S in the figure). → If an observer is present at such an event, then that observer's entire potential future lies within the black hole, not outside it – which means they **will** fall into the black hole, no matter what they do! This "point of no return" is called the event horizon of the black hole. Note that in addition to tipping over, the width of the cone also narrows as it approaches the black hole (even before the event horizon), which means it takes light longer and longer to reach a far-away observer.



Recent observations have indicated a tipping-**apart** effect which becomes significant on cosmological scales, known as the "cosmological constant" or "dark energy" (see GR1f). →The effect does not appear to be related to the presence of any known matter or energy sources, and causes space to expand "all by itself" as time passes.



Four-Force
In general relativity it is easier to talk about the "equation of motion" than "force", which is discussed briefly in the next two sections, and in more detail in GR3x.

Proper Time
In both special and general relativity, time derivatives are taken with respect to the proper time (τ). Since proper time is invariant, this guarantees that the time derivative of any four-vector is also a four-vector. But proper time is defined a little bit differently in general relativity than it is for special relativity (GR1b and GR2b).

Given a coordinate system x^μ and a corresponding metric tensor $g_{\mu\nu}$, the proper time τ between two events along *any timelike* (but not necessarily geodesic) path P(λ) is given by the line integral :

$$\tau = \int_P \sqrt{dx_\mu \, dx^\mu} = \sqrt{g_{\mu\nu} \, dx^\mu \, dx^\iota}$$

which is the arc length of the worldline thru the curved 4-D spacetime. Or, if the path is parameterized by λ :

$$\tau = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \, d\lambda$$

η should be g

From a mathematical point of view, coordinate time is assumed to be predefined and we want an expression for proper time in terms of coordinate time. But when experiments are done, proper time is what is measured (because all clocks are within the frame) and then coordinate time is calculated.

Affine Parameters and Geodesics

When describing a parameterized path thru spacetime such as $x^a(\lambda)$, there is a special class of parameters called **affine** parameters (or sometimes "privileged" parameters). In general, two affine parameters are always linearly related, which means that if *r* and *s* are both affine parameters for the same path, then there exist constants *a* ($\neq$0) and *b* such that r = as+b. Conversely, if *s* is an affine parameter, then r = as+b is also an affine parameter.

Affine parameters are used with geodesics, where the path tangent vector $T^a=dx^a/d\lambda$ is *parallel-transported along its own path*, which means the vector stays parallel to itself and doesn't change length. This is equivalent to saying that the covariant derivative of the path tangent vector is always equal to some constant $\varphi(\lambda)$ times the original tangent vector :

$$\nabla_T T^a = T^b \nabla_b T^a = \varphi(\lambda) T^a$$

$$D(\partial x^\alpha/\partial\lambda)/D\lambda = \varphi(\lambda)\cdot(\partial x^\alpha/\partial\lambda)$$

$$\partial^2 x^\alpha/\partial\lambda^2 + \Gamma^\alpha{}_{\beta\gamma}(\partial x^\beta/\partial\lambda)(\partial x^\gamma/\partial\lambda) = \varphi(\lambda)\cdot(\partial x^\alpha/\partial\lambda)$$

Furthermore, it is always possible to pick a parameter λ such that $\varphi(\lambda)=0$ for all λ, in which case λ <u>is</u> an affine parameter and so

$$\partial^2 x^\alpha/\partial\lambda^2 + \Gamma^\alpha{}_{\beta\gamma}(\partial x^\beta/\partial\lambda)(\partial x^\gamma/\partial\lambda) = 0$$

or equivalently

$$\partial u^\alpha/\partial\lambda + \Gamma^\alpha{}_{\beta\gamma} u^\beta u^\gamma = 0$$

is the **geodesic equation of motion**, which says that the path an object will follow in a gravitational field in the <u>absence of any **non-gravitational** forces</u> (see next section) is a *geodesic*. The choice of λ <u>must</u> be made so that the path tangent vector $dx^a/d\lambda$ has a constant length along the path, but because it is affine any linear transformation of λ (such as $\mu = a\lambda + b$) still satisfies this equation. The ability to linearly scale the parameter just means we can change what the unit of length is (meter vs. light-year), and choose any point as λ=0. Note that since this equation is almost always used for objects, the path is timelike, and so many authors use τ as the parameter in this equation.

Remembering that the length of the tangent vector $T=dx^a/d\lambda$ along a geodesic must be constant :

$$T\cdot T = g_{ab}(dx^a/d\lambda)(dx^b/d\lambda) = (g_{ab}dx^a\, dx^b)/d\lambda^2 = (ds)^2/(d\lambda)^2 = k^2 \text{ (some constant value)}$$

Which means $ds=k\cdot d\lambda$, and so the curved spacetime path distance (ds) between two events along the curve is proportional to the change in the value of the affine parameter (λ) between the events.

For a geodesic, the parameter λ represents time for a timelike curve and distance for a spacelike curve. For timelike paths, proper time τ is the affine parameter for which the length of the path tangent vector is ± 1 or $\pm c^2$ (depending on your units), in which case the path tangent vector represents velocity.

Unfortunately, the length of the tangent vector for a photon travelling along a null geodesic is zero (as is the path's entire spacetime length), so "path length" is meaningless. But all photons have a 4-frequency $\mathbf{k} = (\nu,\nu\mathbf{n})$ and a wavelength. So a reasonable (but not required) affine parameter for null geodesics is to link it to the wavelength count (also called its "phase"). If an observer in a Lorentz frame with local proper time τ measures a photon to have frequency ν, it turns out that the ratio τ/ν is an affine parameter for the photon! Most authors call this λ, but it is not the wavelength of the photon. However, λ is in a sense related to the "length" of the geodesic in that if along a photon's geodesic, event A happens at λ=0 and event B happens at λ=1, then we can say that event C which happens at λ=2 is twice as "far" from A as B is from A.

Since a parallel-transported vector **V** has a constant magnitude (**V**·**V**) along the path, the entire geodesic can be classified as timelike, spacelike or null because the tangent vectors cannot change their character.

Proper Acceleration

In general relativity, **proper acceleration** is the acceleration measured by an accelerometer attached to the object in question (and so is a Lorentz-invariant scalar). In essence, it is the g-forces (F=ma) experienced by the object. This leads to some weird definitions of whether or not an object is "accelerating" :

A rocket firing its engines is accelerating (the passengers feel a g-force)

You, sitting in a chair right now, are also accelerating! This is due to the force of the ground pushing up on you to keep you from moving downwards (you are feeling a g-force of 1-g down, by definition)

Any object falling towards the earth (with little wind resistance) is not accelerating! It is in *free-fall*, and so feels no g-forces (this also includes objects in orbit around massive bodies)

Compared to the definition of free fall in GR1c, we can see that the proper acceleration is zero whenever an object is in free fall. The velocity of a free falling (or orbiting) object is changing relative to the massive body, but this is *ordinary* acceleration – not the same thing, because with a suitable change of reference frames, it goes away. In the same way, the spacetime worldline of such an object can appear to be curved (like an orbit), but by a suitable transformation to a different frame of reference, the path will appear straight. Many authors may be referring to proper acceleration when they say "acceleration" (especially when referring to geodesics), so be careful!

Proper acceleration is the magnitude of the object's 4-acceleration, which in general relativity is given by :

$$A^\alpha = Du^\alpha/D\tau = \partial u^\alpha/\partial\tau + \Gamma^\alpha{}_{\beta\gamma}u^\beta u^\gamma$$

And so when the proper acceleration is zero,

$$\partial u^\alpha/\partial\tau + \Gamma^\alpha{}_{\beta\gamma}u^\beta u^\gamma = 0$$

the object follows a geodesic! Thus if an object is following a geodesic, there are no non-gravitational forces acting on it, but this does not mean its speed is constant!

Recap : Parallel Transport and Geodesics

The concepts of parallel transport and geodesics have been described in several different ways, so it might be a good idea to summarize them all in one place.

A vector **V** *in the tangent plane* has been parallel transported along a path $x^\alpha(\lambda)$ with *path tangent vectors* **T**(λ) when :

- **V** remains parallel to itself and does not change length
- $\nabla_T V^\alpha = DV^\alpha/D\lambda = 0$, which means that the derivatives of the components of **V** with respect to the coordinates cancel the change in the bases due to the curvature of the surface

Properties of parallel transport :

- Two vectors being parallel transported together around the same path remain at the same angle with respect to each other

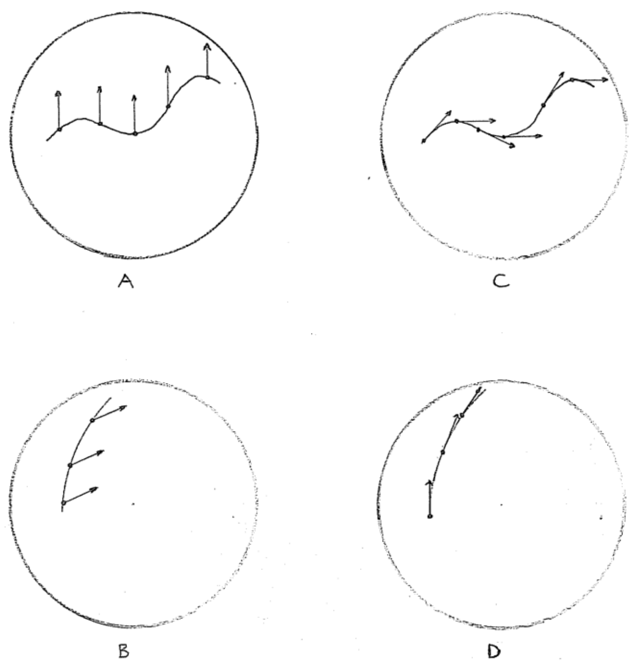
A geodesic is a path which :

- is the shortest spacetime distance in the surface between two points
- is the worldline of a free-falling object
- parallel transports its own path tangent vector : $\nabla_T T^\alpha = 0$
  - so the path tangent vector locally points in the same direction and does not change length
  - if we start with an initial path tangent vector, then we can find the path which keeps its covariant derivative zero, and so trace out a geodesic
- satisfies $\partial u^\alpha/\partial\lambda + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0$, which means the **proper** acceleration is zero at every point along the path
- locally continues to point in the same direction all along the path

And when a path is geodesic :

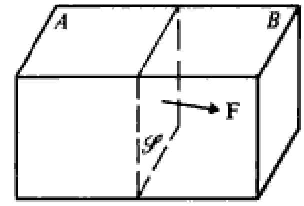
- any tangent vector travelling along it will be parallel transported
- the value of the parameter λ used to describe a geodesic is proportional to the geodesic's path length
- λ also happens to be proportional to the *proper time* if the geodesic is *timelike*
- note that some authors claim that for an object travelling along a geodesic, its speed remains constant (because the "acceleration" is zero), but this statement is frame-dependent!

In the figures of spherical surfaces below, a vector is parallel transported along a wandering curve in A, and along a geodesic in B. The reason the wandering curve isn't a geodesic can be seen in C, where its path tangent vector does not remain parallel to itself along the path. Since the path's tangent vector does stay locally parallel to itself in D, that path is a geodesic.

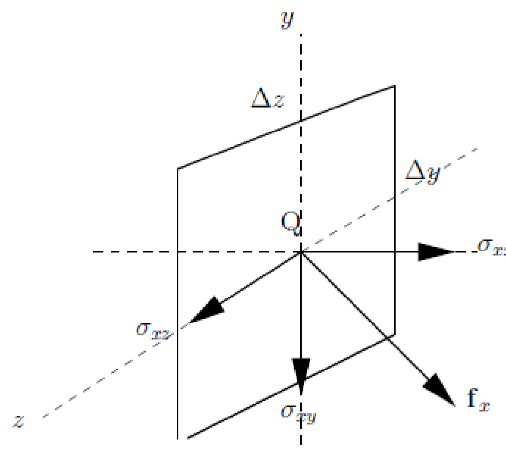


Bottom line : you can always parallel transport any tangent vector along any curve, but only when a curve's *path tangent vector* remains constant when parallel transported along the curve is that curve a geodesic!

Stress-Energy-Momentum (SEM) Tensor



Since a major portion of the SEM tensor comes from the 3-D stress/pressure tensor, let's start with it. The stress tensor measures the internal forces that one part of the material exerts on another part, even tho there may be zero net force at each point, as in the case of equilibrium. If S is a surface element, the material on side A is exerting a force **F** on side B, even in equilibrium. → We thus need to distinguish one side of the surface from another, so we define a normal vector **n** that points in the direction of **F** exerted by the material *behind* the surface. This force can be thought of as that which *would* cause the material in A to flow (accelerate) into B if the material in B were to suddenly vanish. Because the magnitude of the total force depends on the size of the surface, we normalize it : **T** = **F** / S, which is called pressure, stress, shear, etc. Since force equals rate of change of momentum, A would transfer momentum $\Delta p = F\Delta t$ into B over time if the material in A could flow freely.



In this figure, the force $\mathbf{f}_x$ on $S=\Delta y\Delta z$ has three components so that $\mathbf{f}_x = T\cdot S = (\sigma_{xx},\sigma_{xy},\sigma_{xz})\,\Delta y\Delta z$. Note that σ_{xx} points along **n**. Since there are forces on each of the three faces ($\mathbf{f}_x,\mathbf{f}_y,\mathbf{f}_z$) there will be 9 such stress components T_{ij}, with dimensions of force per area. Thus **T** is a rank-2, 3x3 tensor :

$$T_{ij}= \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

In 4-D general relativity, any specific element of the SEM tensor T^{uv} is often described as the "flux of 4-momentum p^u across a surface of constant x^v". Since the standard 3-D definition of flux is the "amount of something that flows through a unit area per unit time", this would mean T^{uv} is the amount of 4-momentum p^u that flows per time across a surface of constant x^v.

First off, a couple of re-definitions are needed. In 4-D spacetime, the "surface" is actually a 3-D "volume" element (and volume is in quotes because one of the dimensions may be time!). The phrase "surface of constant x^v" means that the surface normal **n** is pointing in the x^v direction, or (equivalently) that the surface is perpendicular to the x^v direction, so the "volume" element won't contain x^v.

So there are many problems with the original standard definition above, because :

- Time is one of the coordinates (so "per time" can become a problem)
- In spacetime the surface becomes a "volume" (which may also have a time dimension)
- Sometimes the momentum component is not normal to the surface (so "flows across" is meaningless)

Therefore each term or set of terms must be handled on a case-by-case basis. Note that in the following, superscripts *i* and *j* = 1,2,3 (not time).

Remembering that

$$\mathbf{p} = (mc, mv^1, mv^2, mv^3) = (E/c, mv^1, mv^2, mv^3)$$

Note that the first term of the 4-momentum is mc = E/c. Here it will be called "mass-energy" in quotes to indicate not only their equivalence but also the need for extra scaling factors which aren't visible if c=1.

Measuring the 4-momentum thru the "volume" in a locally inertial tangent space, we have :

$$T^{\mu\nu} = \begin{pmatrix} \frac{\Delta(mc)}{\Delta x\Delta y\Delta z} & \frac{\Delta(mc)}{\Delta y\Delta z\Delta t} & \frac{\Delta(mc)}{\Delta x\Delta z\Delta t} & \frac{\Delta(mc)}{\Delta x\Delta y\Delta t} \\ \frac{\Delta(mv^x)}{\Delta x\Delta y\Delta z} & \frac{\Delta(mv^x)}{\Delta y\Delta z\Delta t} & \frac{\Delta(mv^x)}{\Delta x\Delta z\Delta t} & \frac{\Delta(mv^x)}{\Delta x\Delta y\Delta t} \\ \frac{\Delta(mv^y)}{\Delta x\Delta y\Delta z} & \frac{\Delta(mv^y)}{\Delta y\Delta z\Delta t} & \frac{\Delta(mv^y)}{\Delta x\Delta z\Delta t} & \frac{\Delta(mv^y)}{\Delta x\Delta y\Delta t} \\ \frac{\Delta(mv^z)}{\Delta x\Delta y\Delta z} & \frac{\Delta(mv^z)}{\Delta y\Delta z\Delta t} & \frac{\Delta(mv^z)}{\Delta x\Delta z\Delta t} & \frac{\Delta(mv^z)}{\Delta x\Delta y\Delta t} \end{pmatrix}$$

The T^{00} term is called many things, usually the "mass density" or "energy density" (where it is understood that "density" means over a *spatial* volume). But specifically, it is the amount of "mass-energy" in a 3-D spatial volume at a given instant ("surface of constant x^0" = time).

The T^{i0} terms (left column) are often called "momentum density", and are best described as the amount of 3-momentum (either p_x, p_y, or p_z) in a 3-D spatial volume at a given instant. In a rotating spherical object like a planet or sun, the $T^{30} = T^{\varphi t}$ term would be non-zero, representing the instantaneous *momentum in the φ direction* of the matter within a given volume. If there is no net macroscopic movement of "mass-energy" within the object (rotation, convection, radiation, etc.), these terms are zero.

The T^{0j} terms (top row) are typically called the "energy flux", altho they also go by many names. So T^{0j} is the amount of "mass-energy" flowing over time thru a 2-D area normal to the x^j direction. This could represent movement of actual matter, or energy such as heat transfer. Again, in a rotating sphere the $T^{03} = T^{t\varphi}$ term would be non-zero, representing the *amount of matter moving in the φ direction thru an area ΔrΔθ every second*. And these terms are also zero if there is no macroscopic movement of "mass-energy" within the object.

Some authors claim that the T^{i0} and T^{0j} terms are equivalent and interchangeable, but if you look closely they have different definitions and even different units!

The T^{ii} terms are the amount of 3-momentum p_i flowing over time thru a 2-D area normal to the x^i direction. This is often called "pressure" because :

$T^{xx} = \Delta p / \Delta V = \Delta(mv) / (\Delta t\Delta y\Delta z) = [\Delta(mv)/\Delta t] / (\Delta y\Delta z) = \Delta F / \Delta A$

is the definition of pressure. This makes sense, because pressure differences cause momentum flows (except in equilibrium, which is discussed below).

To figure out what the off-diagonal T^{ij} terms mean (because the momentum component is tangent to the area, so there is nothing "flowing thru" it), let's look at one particular example :

$T^{xz} = \Delta p^x / (\Delta x\Delta y\Delta t) = (F^x\Delta t) / (\Delta x\Delta y\Delta t) = F^x / (\Delta x\Delta y)$

These are often called "shear" or "shear stress", which are forces applied parallel to a surface (σ_{xy} and σ_{xz} in the figure above).

Regarding the "mass-energy" term and considering T^{ab} as a "momentum density"–

Looking back at the original definition of T^{ab}:	$d(mc, mv^x, mv^y, mv^z)^T / dV$
We can factor out a *c* to make T^{ab} a "mass density" :	$c[d(m, m\beta^x, m\beta^y, m\beta^z)^T / dV]$
Or add an extra *c* to make it an "energy density" :	$[d(mc^2, mcv^x, mcv^y, mcv^z)^T / dV]/c$

And all three forms are used by different authors. The *c* term is then incorporated into the constant multiplier of the T^{ab} term on the right-hand side of Einstein's equation which goes unnoticed if c=1. So, if the constant is $8\pi G/c^2$, T^{ab} is mass density. T^{ab} is momentum density if it is $8\pi G/c^3$, and if it is $8\pi G/c^4$ then T^{ab} is energy density. If T^{ab} is a mass density, then any energy terms in T^{00} would contribute E/c^2. Likewise, if T^{ab} is an energy density, then any mass terms in T^{00} would contribute mc^2.

What can we do with the SEM tensor? In a region of space with a particular T^{ab}, an observer with 4-velocity **u** will see a 4-momentum *volume density* (**p**/V) of :

$$p^a/V = -T^a{}_b u^b = -T_b{}^a u^b$$

Along the direction of a unit vector **n**, they will see that component of the 4-momentum volume density :

$$-T_{ab}u^a n^b = -T_{ab}n^a u^b$$

And to get the total mass-energy volume density as measured in their frame :

$$-T_{ab}u^a u^b$$

Or more generally, under a Lorentz transformation $L_a{}^b$, the components T^{cd} in the new frame are $T^{ab}L_a{}^c L_b{}^d$

As we have seen in GR2b and the above, a "mass" term in the metric (such as in the Schwarzschild and Kerr solutions in GR1e) is really the object's **total** mass-energy because it includes the object's rest mass, any electric and magnetic fields it might contain, kinetic energy from macroscopic motion, internal kinetic energy from temperature, internal stresses and pressure, and the binding energy of its particles, atoms, and nuclei.

Representing electromagnetic energy in the SEM tensor has already been discussed in GR2e.

Binding energy is negative – to remove an electron from an atom, you need to add enough energy *E* to overcome the binding energy. Therefore, the sum of the separate masses (ion + electron) is greater than the mass of the atom by E/c^2. Put another way, energy (in the form of a photon) will be given off when the ion and electron are re-combined, and the photon accounts for the (infinitesimally) smaller mass of the combination.

"Gravitational potential energy" (based on height above a surface) definitely does not contribute to the SEM tensor, because it does not exist in general relativity, just like "gravitational force" no longer exists. But the question of whether a gravitational field itself further curves spacetime is complicated (it is argued both ways), and intertwined with the concept of conservation of energy in general relativity, *which apparently is still not well understood.*

It should be noted that while both mass and energy contribute to the SEM tensor, it is mass that dominates due to the c^2 term in $E=mc^2$. For example, the amount of energy the sun puts out every second is only $\sim 10^{-24}$ of its mass! And the rotational energy of the entire galaxy is only $\sim 10^{-6}$ of its mass. So if there are masses involved, we can usually ignore the other energies.

The divergence of the stress-energy-momentum tensor

$$T^{\mu\nu}{}_{;\nu} = \nabla_\nu T^{\mu\nu} = 0 \qquad \text{or} \qquad \nabla^\mu T_{\mu\nu} = 0 \quad \text{(note change of "wrt index"!)}$$

leads to the local inertial frame (= flat spacetime) conservation of energy and conservation of momentum.

For example, in a locally flat spacetime, this reduces to : $\partial_\nu T^{\mu\nu} = 0$ where for $\mu=0$ we have

$$\partial T^{0\nu}/\partial x_\nu = 0$$

$$\partial T^{00}/\partial x_0 + \partial T^{01}/\partial x_1 + \partial T^{02}/\partial x_2 + \partial T^{03}/\partial x_3 = 0$$
$$\partial E'/\partial t + \mathbf{\nabla}\cdot\mathbf{p}' = 0$$
$$\partial E'/\partial t = -\mathbf{\nabla}\cdot\mathbf{p}'$$

Where E' is the "mass-energy" volume density and **p**' is the 3-momentum volume density. Integrating over space to remove the densities, this equation says that the rate of change of "mass-energy" in a volume over time must equal the flow of "mass-energy" in or out of the volume, which is "conservation of energy". The three spatial components (μ=1,2,3) lead to an equation describing the connection between the forces acting on the volume and the stresses within it.

A "**perfect fluid**" has no viscosity, heat flow, or shear, and (if it is not rotating) can be described completely by its rest-frame mass density ρ and its rest-frame pressure P (both of which are Lorentz invariant scalars). If u^μ is the 4-velocity of the fluid's particles (or an observer viewing them, depending on one's frame of reference), the *metric-free* and *coordinate-free* mixed-type SEM has the form

$$T^\mu{}_\nu = (\rho+P/c^2)u^\mu u_\nu - P\delta^\mu{}_\nu$$ (note : $u^\mu u_\nu$ is a dyadic product!)

so that in the comoving rest frame, where $u^\mu = (c,0,0,0)$ and $u^\mu u_\nu = \mathrm{diag}(c^2,0,0,0)$ it becomes

$$\begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix}$$

and note that this SEM tensor is in the form of an energy density. Using the metric to lower the contravariant index, the SEM becomes

$$T_{\mu\nu} = (\rho+P/c^2)u_\mu u_\nu - Pg_{\mu\nu}$$

so that in the comoving rest frame, where $u^\mu = (c,0,0,0)$ and $g_{\mu\nu}= \mathrm{diag}(1,-1,-1,-1)$:

$$T_{\mu\nu} = (\rho+P/c^2)\cdot\mathrm{diag}(c^2,0, 0, 0) - P\cdot\mathrm{diag}(1,-1,-1,-1) = \mathrm{diag}(\rho c^2,P,P,P)$$

or

$$\begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Note that some authors describe T_{uv} as

$$\begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & Pg_{11} & 0 & 0 \\ 0 & 0 & Pg_{22} & 0 \\ 0 & 0 & 0 & Pg_{33} \end{pmatrix}$$

but they are just not converting to the comoving rest frame of the fluid (so the g_{ii} terms are the terms of the metric in whatever frame they are using).

How can "pressure" create an attractive force? When we usually think of pressure we might think of the expansion of a balloon, but this is due to a pressure difference, which also creates the wind and ear-popping when you fly or dive deep underwater. But even going by the definition that the T^{ii} terms are related to "3-momentum flowing thru a 2-D area", how can there be any internal momentum (and hence pressure) when the object is in *equilibrium*?

To fully understand what the pressure term represents, we must look at the material of the object at a microscopic level. While there is no macroscopic movement of the material in equilibrium, the molecules of the material are moving, and are constantly bouncing off the sides of the (virtual) box originally described above. Each bounce transfers some momentum from the molecule to the side of the box, which creates the pressure. A simple model results in an expression for pressure of

$$P = mv^2/3V$$

where *m* is the total mass of the material in the box, *v* is the average velocity of the molecules, and *V* is the volume of the box.

What this means is that the kinetic energy due to the motion of the molecules is creating the pressure. Since kinetic energy $E_K = mv^2/2$, we can solve for v and substitute this into the previous formula to get :

$$P = 2E_K/3V$$

which is an **energy density**!

Thus the "pressure" term *P* is really just proportional to the internal kinetic energy of the molecules in the material, and *any energy is going to cause spacetime to curve.* This makes sense in two other ways : in gasses, pressure increases as temperature (molecular kinetic energy) increases; and "dust" (below) is defined as having zero pressure and zero relative movement between particles.

However, we seldom notice this in every-day life because the pressure contribution is a factor of c^2 smaller than the mass-energy density term. For example, the average density of the sun is about 1400 kg/m^3 and its average pressure is about $2.5*10^{16}$ N/m^2, but density*c^2 = $1.26*10^{20}$, about 20,000 times bigger! Only in a neutron star does the pressure begin to be comparable to the mass-energy density.

In general relativity, "dust" is a special case of a perfect fluid which has zero pressure : $T_{\mu\nu} = \rho u_\mu u_\nu$
Its particles either do not interact or only interact gravitationally (there are differing definitions). But they definitely do not collide with each other, and they have zero relative motion with respect to each other (otherwise they would eventually "bounce off the walls of the box", creating pressure). This is confirmed by $\nabla_\nu T^{\mu\nu} = 0$, which (with some math) shows that the acceleration of the particles is zero.

Perfect fluids are used extensively in cosmology – the universe is modeled as a time-varying dust (with each particle being an entire galaxy!), and the interior of stars (including white dwarfs and neutron stars) are considered as non-rotating perfect fluids. Note that assuming the pressure and density are constant thruout the object (the Schwarzschild interior solution) is highly unrealistic.

It seems that the SEM tensor is still not well understood. Einstein himself wrote,

> …the energy tensor can be regarded only as a provisional means of representing matter. In reality, matter consists of electrically charged particles... It is only the circumstance that we have no sufficient knowledge of the electromagnetic field of concentrated charges that compels us, provisionally, to leave undetermined in presenting the theory, the true form of this tensor. From this point of view it is at present appropriate to introduce a tensor, T_{uv}, of the second rank of as yet unknown structure, which provisionally combines the energy density of the electromagnetic field and that of ponderable matter; we shall denote this in the following as the "energy tensor of matter".... The right side [of his equation] is a formal condensation of all things whose comprehension in the sense of a field-theory is still problematic. Not for a moment, of course, did I doubt that this formulation was merely a makeshift in order to give the general principle of relativity a preliminary closed-form expression. For it was essentially not anything *more* than a theory of [just] the gravitational field, which was isolated somewhat artificially from a total field of as yet unknown structure.

Neither Einstein nor anyone since has been able to make further progress in determining the true form of the SEM tensor, although it is at the heart of current efforts to reconcile quantum mechanics with general relativity.

Energy Conditions

Because general relativity does not provide global energy conservation and puts no bounds on the relationships between the components of the SEM (other than state equations), the following energy conditions are various criteria that help reject solutions that "shouldn't" be physically possible.

In terms of a perfect fluid SEM, the different kinds of energy conditions can be stated as :

Dominant :	$\rho c^2 \geq \lvert P \rvert$	(implies $\rho c^2 \geq 0$)
Strong :	$\rho c^2 + P \geq 0$	$\rho c^2 + 3P \geq 0$
Weak :	$\rho c^2 + P \geq 0$	$\rho c^2 \geq 0$
Null :	$\rho c^2 + P \geq 0$	
Trace :	$\rho c^2 - 3P \geq 0$	

Remember that ρ and P refer to the mass density and pressure density in the rest-frame of the fluid. In a different frame of reference, the ρ and P parts of the SEM tensor would mix, and it might be possible to end up with a negative density. The Weak energy condition guarantees that the mass-energy density is never negative in any frame, and the Dominant energy condition guarantees *in addition* that no observer will see mass-energy moving with a velocity greater than *c*. The Trace and Strong conditions have been abandoned as unrealistic.

However, it should be pointed out that there are some known physical examples at both quantum and cosmological scales involving negative energy densities and negative pressures that violate some of these conditions. The Casmir effect (the attraction between two uncharged metal plates due to quantum vacuum energy) and cosmological dark energy (see GR1f) are two examples. So the status of these conditions is currently under debate.

Weyl Tensor

As we saw in GR1c, if we take a small ball of particles and move them along geodesics, the ball can change its volume and can also change its shape to an ellipsoid. The 10 independent components of the Ricci tensor describe how the volume of the ball starts changing in any given direction, while the Weyl tensor describes how it changes its shape. So there are two independent types of curvature : Ricci curvature and Weyl curvature. The complete curvature of spacetime is a combination of these two.

(i) If the Ricci tensor is zero but Weyl tensor is not, the ball will be changed into an ellipsoid without changing its volume ("tidal forces").
(ii) If the Weyl tensor is zero but the Ricci tensor is not, the ball will only change its volume but not its shape.

(i) (ii)

The source of Ricci curvature is the stress-energy-momentum of *local* (**<u>within the region</u>** we are solving for) matter-energy. If the matter-energy distribution is zero, then the Ricci tensor will be zero. But spacetime is not necessarily flat in this case, since the Weyl tensor can contribute to curvature. The Weyl tensor allows matter at one point to have a gravitational influence on distant places. Thus, the Weyl tensor represents spacetime curvature which can propagate thru a vacuum, such as gravitational waves and tidal forces. Another way to describe it is that the Weyl tensor measures the <u>deviation</u> of a surface from being *conformally flat*, where the metric $g = \lambda^2(x^a)\cdot\eta$.

Because the SEM tensor defines the Ricci tensor, determining the Weyl tensor (distant influences) and the SEM tensor (sources of local influence) pretty much determines g_{uv}.

The Schwarzschild and Kerr solutions (GR1e), which model spacetime <u>**outside**</u> a static or rotating body, have zero Ricci curvature but non-zero Weyl curvature everywhere. The Friedmann dust solution, which models the universe on a very large scale (GR2g), has zero Weyl curvature but nonzero Ricci curvature at each point in spacetime. Any spacetime with <u>constant</u> curvature also has zero Weyl curvature.

In terms of the Riemann, Ricci and metric tensors, the Weyl tensor C is given by :

$$C_{\kappa\lambda\mu\nu} \equiv R_{\kappa\lambda\mu\nu} - \tfrac{1}{2}\left(g_{\kappa\mu}R_{\lambda\nu} - g_{\kappa\nu}R_{\lambda\mu} + g_{\lambda\nu}R_{\kappa\mu} - g_{\lambda\mu}R_{\kappa\nu}\right) + \tfrac{1}{6}\left(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}\right)R$$

Since the Riemann and Ricci tensors and the Ricci scalar are all related to the metric as we have seen in GR1d, it would also be possible (but extremely messy) to express the Weyl tensor just in terms of g_{uv}.

The equation that relates the Weyl tensor to the distortion of spacetime far from the source creating it is :

$$\nabla^\alpha C_{\alpha\sigma\mu\nu} = g^{\alpha\beta}\nabla_\beta C_{\alpha\sigma\mu\nu} = 8\pi G[\nabla_\mu T_{\nu\sigma} - \nabla_\nu T_{\mu\sigma} + g_{\sigma\mu}\nabla_\nu T^\alpha{}_\alpha/3 - g_{\sigma\nu}\nabla_\mu T^\alpha{}_\alpha/3] / c^4$$

<u>**This equation and Einstein's equation together define the spacetime curvature at any point in spacetime caused by any mass-energy distribution.**</u>

Given direction vectors **k**, **m**, and **n** (where <u>**m**</u> is the complex conjugate of **m**), the ten independent components of the Weyl tensor can be determined from five possibly complex (!) scalars :

$$\Psi_0 = C_{\kappa\lambda\mu\nu}\, k^\kappa\, m^\lambda\, k^\mu\, m^\nu$$
$$\Psi_1 = C_{\kappa\lambda\mu\nu}\, k^\kappa\, n^\lambda\, k^\mu\, m^\nu$$
$$\Psi_2 = C_{\kappa\lambda\mu\nu}\, k^\kappa\, m^\lambda\, \underline{m}^\mu\, n^\nu$$
$$\Psi_3 = C_{\kappa\lambda\mu\nu}\, n^\kappa\, k^\lambda\, n^\mu\, \underline{m}^\nu$$
$$\Psi_4 = C_{\kappa\lambda\mu\nu}\, n^\kappa\, \underline{m}^\lambda\, n^\mu\, \underline{m}^\nu$$

These components have the following general physical interpretations :

- Ψ_0 is a transverse component (like water waves) moving in the **n** direction
- Ψ_1 is a longitudinal component (like sound waves) in the **n** direction
- Ψ_2 is a Coulomb-like (point-source radial field) component, responsible for tidal forces
- Ψ_3 is a longitudinal component in the **k** direction
- Ψ_4 is a transverse component moving in the **k** direction

And the only non-zero component of the Weyl tensor for the Schwarzschild solution is : $\Psi_2 = -m / r^3$
For the Kerr solution, it is also the only non-zero component, but it is *complex* : $\Psi_2 = -m / (r - i a\cos\theta)^3$

The Weyl tensor can be separated into "electric" and "magnetic" components (not to be confused with the "gravitoelectric" and "gravitomagnetic" components due to gravitational <u>sources</u> discussed in GR1e), each of which is a rank 2 tensor relative to the observer's velocity ***u***:

$$E_{ab} = C_{acbd}u^c u^d \qquad H_{ab} = \tfrac{1}{2}C_{acst}\eta^{st}{}_{bd}u^c u^d$$
$\eta \rightarrow \varepsilon$

The electric component is responsible for tidal forces and geodesic deviation (next section), but the magnetic component has no equivalent concept in Newtonian physics. The magnetic part is generated by vorticity, shear, and/or rotation. There are no known solutions of Einstein's equation for a purely magnetic gravitational wave travelling thru a vacuum or possibly a dust-filled universe (there seem to be conflicting opinions on the latter). The Weyl tensor is independent of the velocity of the observer, but the electric and magnetic components are not, and they are orthogonal to the observer's direction of motion. But it should be noted that *the Weyl tensor in general, and its components (especially the magnetic one), are still not well understood.*

The Geodesic Deviation Equation

Another way to look at spacetime curvature is to examine the **geodesic deviation**. We know that on a curved surface, geodesics that are parallel at one point get angled and change the distance between them as we advance along them. → They get closer on a sphere or ellipsoid and farther apart on a hyperboloid (a saddle-like surface). Any place on a curved surface can be locally described as ellipsoid, flat, or hyperboloid.

Start with two clocks comoving next to each other in outer space, one foot apart from each other. They are in free fall, so their paths trace out geodesics which may converge, diverge, or remain parallel. A spacetime diagram of the situation is shown below – let **v** be the velocity vector tangent to the worldline of the first clock at point P at time zero, and **s** be the vector from the first clock to the second, which is at point Q at time zero :

```
    v ^        ^ this velocity is the result of parallel transporting v to Q
      |        |    (because they are comoving)
      | s      |
      P->------Q
   clock1    clock2
```

Technically, **s** is a **connecting vector** or **deviation vector**, which points from one geodesic to a neighboring geodesic, and joins two points that have the *same affine parameter value*.

Let each clock wait a second, so they have new positions (in spacetime) P' and Q' :

```
        P'->----Q'
        |       |
        |       |
      v ^       ^
        |       |
        | s     |
        P->-----Q
```

Now, what is the actual velocity vector of the clock at Q'? First we parallel transport **v** along **s** over to Q (because they were initially comoving), then we parallel transport the result over to Q' (which is along a geodesic).

We want to know if the clock at Q' is moving towards or away from the clock at P'. To answer this, we compare the actual velocity vector at Q' (from above) to the vector we get by first parallel transporting **v** over to P', and then parallel transporting again over to Q'. That's the velocity the clock at Q' *would have* if it stayed at rest relative to the clock at P'.

Note that when we do this, we are taking the vector **v** and parallel transporting it two different ways from P to Q' and getting two (possibly) different answers. If the answers were the same, the second clock would remain at rest relative to the first. But if they are not, then the difference tells us how the second one has begun accelerating towards or away from the first.

Now remember how the Riemann tensor works : the result of

(parallel transporting **v** from P to Q [along *s*] to Q') – (parallel transporting **v** from P to P' to Q')

is going to be just

– R(s,v,v)

which is the same as

R(v,s,v)

since the Riemann tensor is defined so that it's anti-symmetric in the first two slots.

In short, the geodesic deviation equation says the following : two initially comoving particles in free fall will accelerate *relative to one another* in a manner determined by the <u>total</u> (= Riemann) curvature of space. Suppose the velocities of two particles are **v**, and the initial displacement **s** from it to the second particle is small. Then the acceleration a^a of the <u>distance between them</u> is given by R(v,s,v). Or with indices :

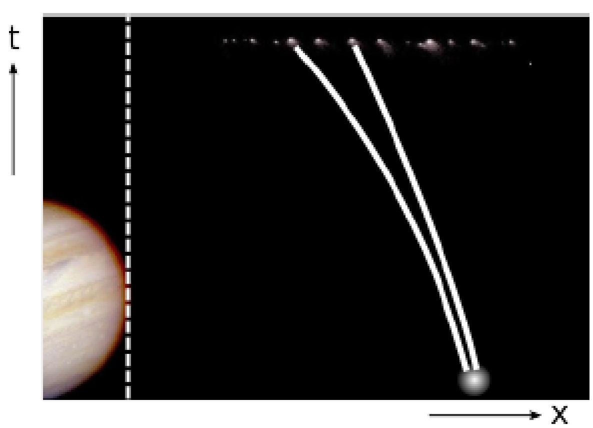
$$a^a = D^2s^a/D\tau^2 = R^a{}_{bcd}\, v^b\, s^c\, v^d$$

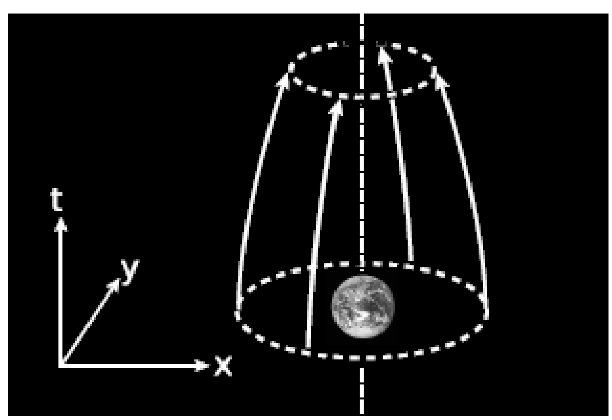
This equation can have many formats due to the many ways to write a parameterized covariant derivative. A factor of –1 may also appear on the right due to the *metric signature* and/or the order of the **v** and **s** vectors (because of the Riemann tensor's anti-symmetry). Also note that <u>**the equation given above is an approximation**</u>, assuming a very small initial distance between the geodesics and then ignoring all second-order terms!

Geodesic deviation is related to the electric portion of the Weyl tensor :

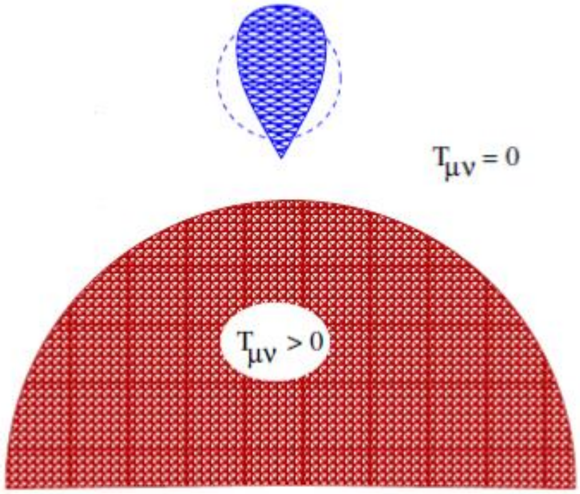
$$a_a = R_{abcd}\, v^b\, s^c\, v^d$$
$$a_a = C_{abcd}\, v^b\, s^c\, v^d \qquad \text{(in a vacuum, } R_{abcd} = C_{abcd} \text{ because Ricci = 0)}$$
$$a_a = s^c\, (C_{abcd}\, v^b\, v^d)$$
$$a_a = s^c\, E_{ac}$$

If two objects are in a Schwarzschild spacetime (outside a large mass) with only Weyl curvature, their geodesics may diverge or converge depending on the initial conditions :





And a solid body near a large mass would be stretched out radially while being pinched in sideways, and more so closer to the large mass :



If you were to fall towards a non-rotating black hole, the tidal forces of geodesic deviation would start to stretch you apart. The "separation" in this case is the distance from your head to your feet (~2 m), and the force would be the acceleration calculated from the formula times your mass (~80 kg). Simplifying the equation and putting back in the factors *G* and *c*, the tidal force at a distance *r* from the singularity is roughly $GmhM/r^3$, where M is the mass of the black hole. That turns out to be about $10^{-8}M/r^3$, and assuming a "typical" black hole about 10 times the mass of the sun, you would feel virtually nothing if you were as far away from it as Mercury is from the sun. But you would feel as if you had a 50 pound weight hanging from your feet at a distance of about 20,000 km (12,000 miles or the diameter of the earth). You would probably black out around17,000 km due to blood being forced to your head, which would be a blessing because your body would be ripped apart around 4000 km away, <u>long before you reached the event horizon</u>! Close enough to the singularity, *even atoms and nuclei are torn apart*.

<u>Metrics and Line Elements and c (oh my!)</u>

As mentioned in GR1b, many authors describe a spacetime by its line element while calling it "the metric". You must be careful when converting a given line element into its metric format, and unfortunately this process is complicated by the standard convention of not showing *c* in the equations. For example, the line element for the Schwarzschild exterior metric is :

$$ds^2 = [1-(2m/r)]\cdot \boldsymbol{dt^2} - 1/[1-(2m/r)]\cdot \boldsymbol{dr^2} - r^2\cdot \boldsymbol{d\theta^2} - r^2\sin^2\theta\cdot \boldsymbol{d\varphi^2}$$

Which when *c* is explicitly shown becomes :

$$ds^2 = [1-(2m/r)]\cdot \boldsymbol{c^2dt^2} - 1/[1-(2m/r)]\cdot \boldsymbol{dr^2} - r^2\cdot \boldsymbol{d\theta^2} - r^2\sin^2\theta\cdot \boldsymbol{d\varphi^2}$$

Everything in bold are elements of the coordinate system (spherical in this case), so <u>everything else</u> goes in the metric. But note that the c^2 comes from the *cdt* (or *dct*) coordinate term which converts time into length, so <u>the *c* does not go into the metric</u>. Thus the above line element represents the following metric g_{uv} :

1–(2m/r)	0	0	0
0	– 1/[1–(2m/r)]	0	0
0	0	$-r^2$	0
0	0	0	$-r^2\sin^2\theta$

Many authors incorrectly include the c^2 term in the metric. This is so that c "carries thru" into the formulas for Christoffel symbols and the Riemann and Ricci tensors, etc. However, when taking derivatives of the metric to build these terms, the derivative with respect to the time coordinate (cdt) divides the result by c. Putting a c^2 term in the metric (and then not dividing by c when taking the derivative) does **not** always give the same results!

The arguments for not including a c^2 term in the metric (or a c term in any g_{u0} or g_{0v} equations) are many – for example, if we define two equivalent velocity vectors in a Minkowski spacetime :

$v^i = [c,0,0,0]^T$ $\qquad\qquad$ $v_i = [c,0,0,0]$

Note that $v^i v_i = c^2$ without the use of a metric, and must remain invariant. The only way that the dot products of the contravariant and covariant vectors with themselves remains the same is to use $g_{ii} = \text{diag}([1, -1, -1, -1])$:

$g^{ii} v_i v_i = c^2$ $\qquad\qquad$ $g_{ii} v^i v^i = c^2$

No other combination of placing c in the vectors (or not) or in the metric (or not) is consistent.

In addition, for a perfect fluid where c is not shown :

$$T^\mu{}_\nu = (\rho+P)u^\mu u_\nu - \delta^\mu{}_\nu P$$

Since in this case there is *no metric*, $(\rho+P)$ must be $(\rho+P/c^2)$ and u^u must be $(c,0,0,0)$ for the units to work out, so the covariant format becomes :

$$T_{\mu\nu} = (\rho+P/c^2)u_\mu u_\nu - g_{\mu\nu}P$$

So that in the comoving rest frame, $g_{\mu\nu}$ **must be** diag(1,–1,–1,–1) to get the right result :

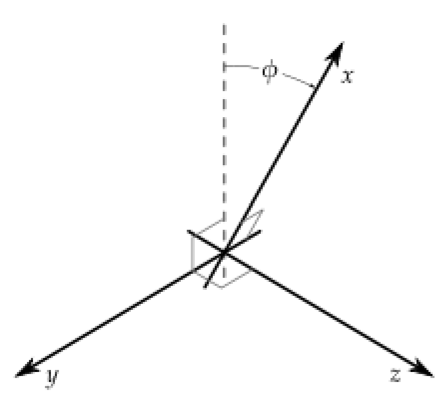
$$T_{\mu\nu} = (\rho+P/c^2)\cdot\text{diag}(c^2,0,0,0) - P\cdot\text{diag}(1,-1,-1,-1) = \text{diag}(\rho c^2,P,P,P)$$

Off-diagonal Metric Terms

While we're talking about metrics, let's look at the Kerr metric :

$1 - 2mr/\Sigma$	0	0	$(2mra/\Sigma)\sin^2\theta$
0	$-\Sigma/\Delta$	0	0
0	0	$-r^2 - a^2\cos^2\theta$	0
$(2mra/\Sigma)\sin^2\theta$	0	0	$-(r^2 + a^2 + (2mra^2\sin^2\theta)/\Sigma)\sin^2\theta$

What do the non-zero off-diagonal terms represent? Off-diagonal terms in the metric means that that pair of basis vectors is not orthogonal. The simplest example of this is 3-D Cartesian coordinates where (for example) the x axis has been tipped by an angle φ relative to the z axis while staying orthogonal to the y axis. →



If $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are unit vectors along the x, y, and z axes, their dot products give the following components of the 3-D metric tensor :

1	0	$\sin\varphi$
0	1	0
$\sin\varphi$	0	1

(note that $\sin\varphi$ = cos(the angle between x and z)

The angle between the axes is easy to determine : using t and φ from the Kerr metric, then by definition :

$$g_{t\varphi} = \mathbf{e}_t \cdot \mathbf{e}_\varphi = e_t\, e_\varphi \cos\alpha$$

where α is the angle between the axes. But also by definition :

$$g_{tt} = \mathbf{e}_t \cdot \mathbf{e}_t = e_t\, e_t$$

so $e_t = \sqrt{g_{tt}}$ and likewise $e_\varphi = \sqrt{g_{\varphi\varphi}}$ thus the angle between the axes is given by :

$$\alpha = \cos^{-1}(g_{t\varphi}(g_{tt}g_{\varphi\varphi})^{-1/2})$$

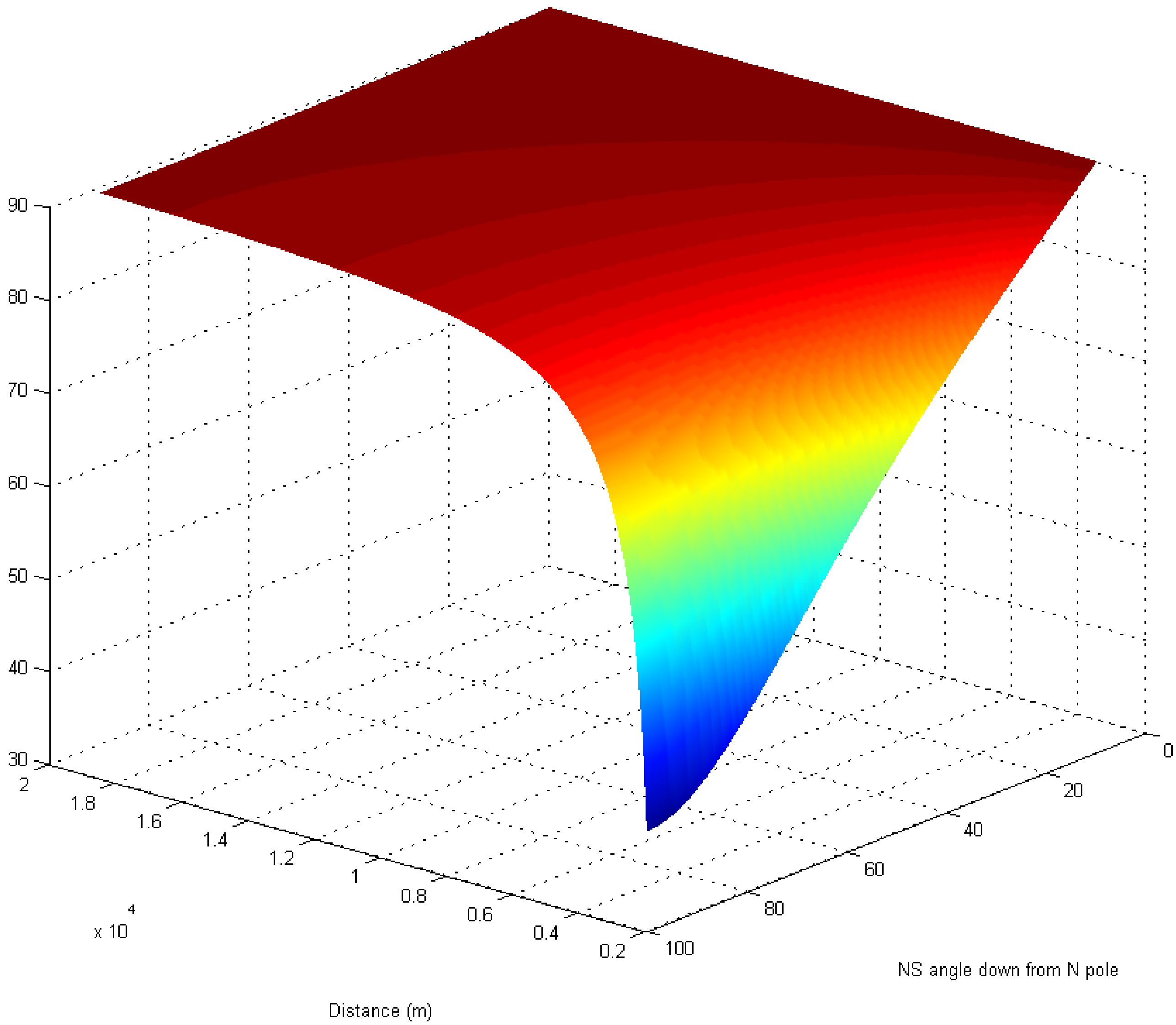
In general, the angle between any pair of off-diagonal axes *u* and *v* is :

$$\cos^{-1}(g_{uv}(g_{uu}g_{vv})^{-1/2})$$

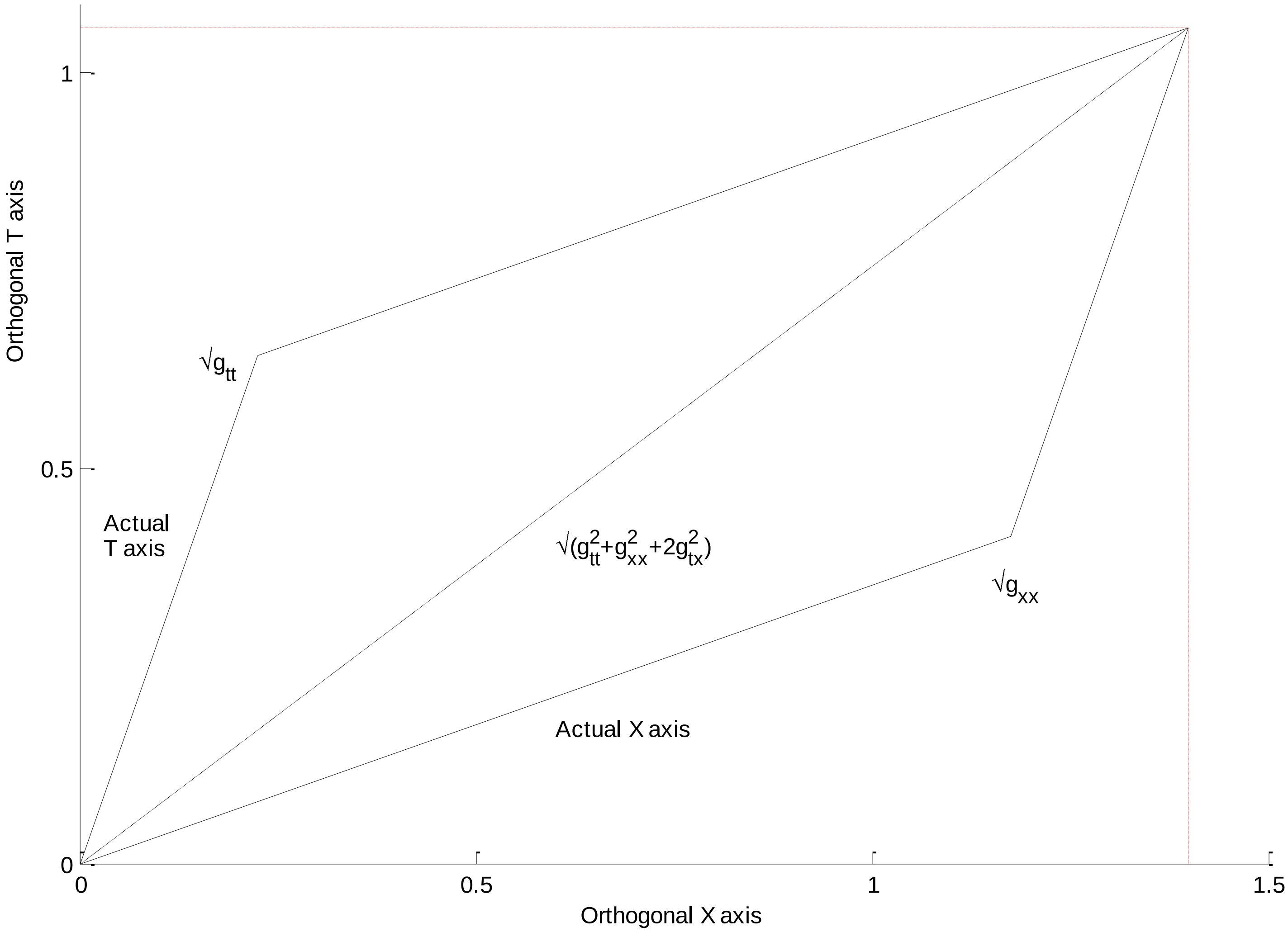
Off-diagonal components only show up when mass-energy moves, such as in the Kerr spacetime or the "moving mass" spacetime shown in GR1e.

In the Kerr spacetime, the time and φ (east-west) coordinates are not perfectly orthogonal, but the angle between them is usually very close to 90° except near the event horizon at the equator when the angular momentum is high. For example, the t-φ angle at the surface of the earth is about 89.999999999999995 degrees.

The figure below shows the t-φ angle as a function of distance and north-south angle around a rotating object with very high angular momentum.

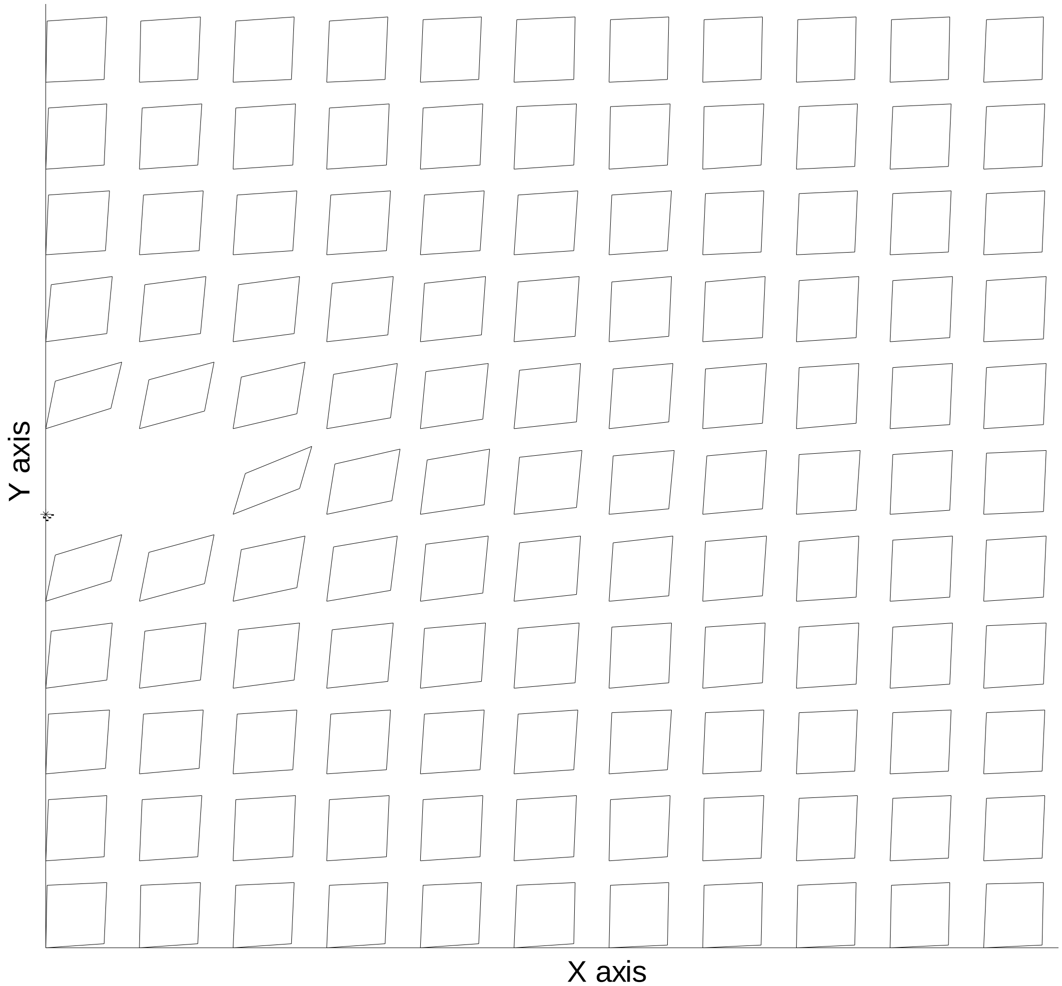


The following figure shows the t-axis stretch, the x-axis stretch, and how the t-x term defines the angle between them for a moving mass. When the axes aren't orthogonal, the diagonal length is $ds^2 = g_{tt}^2 + g_{xx}^2 + 2g_{tx}^2$. As can be seen, the result is that the amount of "effective" stretch projected on to the orthogonal t-x axes (red dotted lines) is even larger than the actual stretch along the actual (tipped) axes.



Orthogonal T axis
1
0.5
0
√g_tt
Actual
T axis
√(g_tt^2+g_xx^2+2g_tx^2)
√g_xx
Actual X axis
0
0.5
1
1.5
Orthogonal X axis

The next figure shows the amount of t-x stretching and angling at different places in the x-y plane of a mass moving in the +x direction at 0.8c. Far from the mass, the axes become close to orthogonal.



Bottom line : diagonal terms in the metric represent the amount that spacetime is "stretched", while off-diagonal terms represent the amount that the axes of spacetime are "angled".