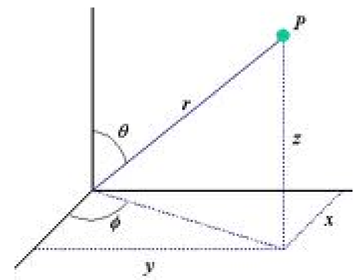
Exact Solutions to Einstein's Field Equation

Einstein's equation can be greatly simplified by assuming that space is empty, or uniformly filled with dust (matter with no pressure), or contains nothing but a single concentrated mass. This last example was solved within weeks after Einstein published his equation, by a German physicist named Schwarzschild.

In spherical coordinates ($0=t$, $1=r$, $2=\theta$, $3=\varphi$), Schwarzschild's $g_{\mu\upsilon}$ for a non-rotating object turns out to be :

$1-(2m/r)$	0	0	0
0	$-1/[1-(2m/r)]$	0	0
0	0	$-r^2$	0
0	0	0	$-r^2\sin^2\theta$



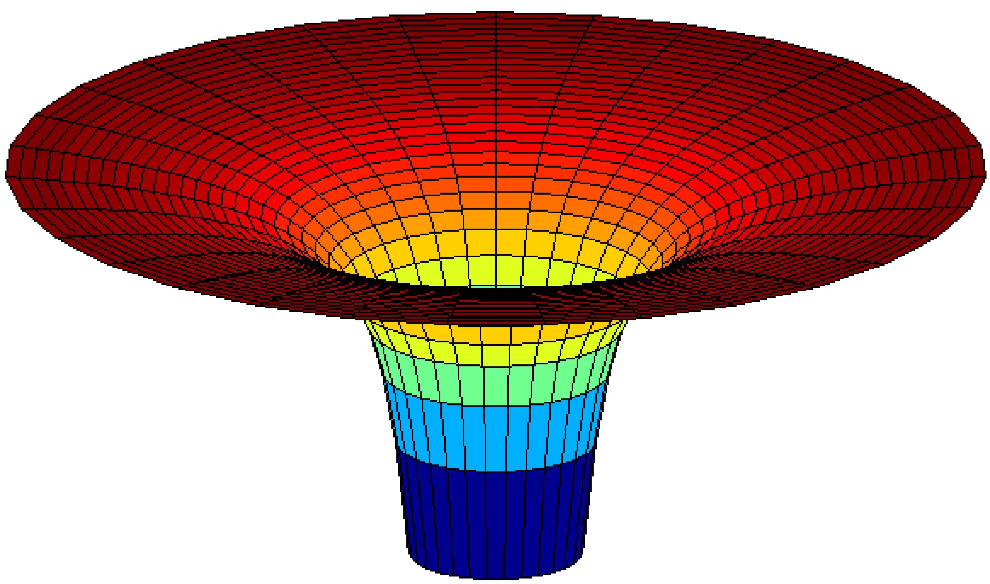
Where

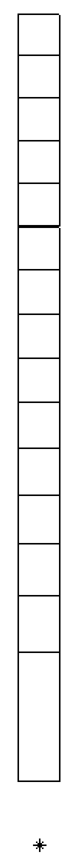
m = the mass of the object

So the line element is :

$$ds^2 = [1-(2m/r)]\cdot \boldsymbol{dt^2} - 1/[1-(2m/r)]\cdot \boldsymbol{dr^2 - r^2\cdot d\theta^2 - r^2\sin^2\theta\cdot d\varphi^2}$$

Everything in bold is the "flat" description of spacetime in spherical coordinates, so what gets stretched is the radial direction (*dr*) and time (*dt*). Looking at the *dr* term, it gets bigger as *r* gets smaller (closer to the mass). So the shape of a 2-D plane of space thru the equator of the mass, at a fixed instant of time, looks like this :





What the color image is saying is that space gets stretched radially as you get closer to the mass. Another way of describing this is with the black and white image to the right, which shows one "strip" of space extending radially out from the mass (the * at the bottom). At the top of the strip, far from the mass, space is not warped at all, as indicated by the square patch. As you move closer to the mass, space gets longer and longer in the radial direction, but always stays the same width. If you repeated this one strip by rotating it in a circle around the mass, you'd end up with the color 3-D picture, where the "extra" space from the stretching is going down to "make room" for it.

Unfortunately, while the 3-D picture is visually more attractive, it is very misleading, in that it looks like the 2-D space is stretching in a direction "outside" of itself (that is, it looks like it is stretching in a direction perpendicular to the space itself), **when it is not**! The flat strip is really more accurate, because it shows that the 2-D space gets stretched "within" the space itself (because the stretching stays within the 2 dimensions of the page that the strip is in). So when our 3-D space gets stretched due to some mass, the stretching does not "leave" our universe, nor does our universe extend into some 4^{th} spatial dimension to "make room" for the stretching.

Also keep in mind that the amount of stretching is <u>very local</u> – for a small object with the mass of our sun, space is essentially flat (less than 1% distortion) by the time you move just 90,000 miles away, or 1/400 of Mercury's orbit.

Looking at the *dt* term, it gets smaller the closer you get to the mass, so that <u>time slows down</u>. Also note that the *dt* term times the *dr* term is a constant, so that *the stretching of space and the slowing of time exactly balance each other out*!

For a <u>spinning mass</u> (like the sun or earth) with $\Lambda = 0$, $g_{\mu\upsilon}$ is a little more complicated :

$1 - 2mr/\Sigma$	0	0	$(2mra/\Sigma)\sin^2\theta$
0	$-\Sigma / \Delta$	0	0
0	0	$-r^2 - a^2\cos^2\theta$	0
$(2mra/\Sigma)\sin^2\theta$	0	0	$-(r^2 + a^2 + (2mra^2\sin^2\theta)/\Sigma)\sin^2\theta$

Where

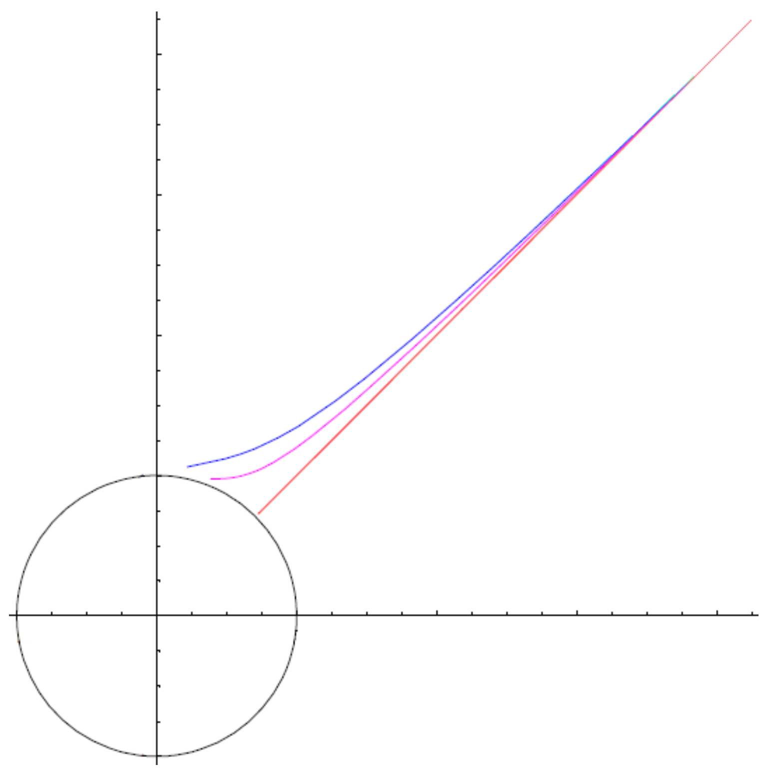
a = the angular momentum of the spinning mass
$\Sigma = r^2 + a^2\cos^2\theta$
$\Delta = r^2 - 2mr + a^2$

So the line element is :

$$ds^2 = [1 - 2mr/\Sigma]\cdot \boldsymbol{dt^2} - [\Sigma/\Delta]\cdot \boldsymbol{dr^2} - [1 + (a^2\cos^2\theta)/r^2]\cdot \mathbf{r^2}\cdot \boldsymbol{d\theta^2}$$

$$- [1 + a^2/r^2 + (2ma^2\sin^2\theta)/(r\Sigma)]\cdot \mathbf{r^2\sin^2\theta}\cdot \boldsymbol{d\varphi^2} + [(4ma/\Sigma)\sin\theta]\cdot dt\cdot(r\cdot\sin\theta\cdot d\varphi)$$

This was solved by a New Zealand physicist named Kerr, <u>47 years</u> after Einstein published his formula (just adding the spin made the problem that much more difficult!). Again, everything in bold is part of the "flat" description of spacetime in spherical coordinates. And if we set $a = 0$, we get the Schwarzschild solution.

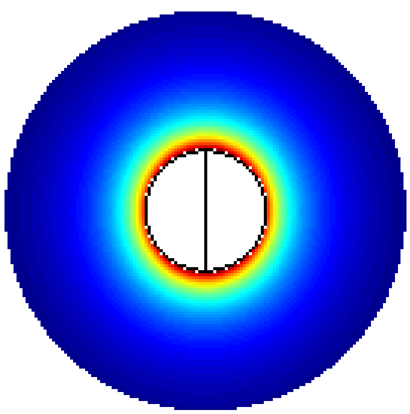


Unfortunately, the way space stretches around a spinning mass can't be shown with a 3-D perspective picture, because instead of stretching just radially, space also stretches *circularly* around the mass (the $d\varphi$ term). It's as if the spinning mass "grabs hold" of space, and stretches it in the direction of rotation ("**frame-dragging**"). This means that an object dropped from far away *straight* towards the center of the mass would pick up some angular momentum (be kicked off to one side) <u>just from the bending of spacetime</u>! → The magenta line is for our sun, the blue line is for a faster spinning star. Newton's law of gravity says that it would just fall straight in, like the red line.

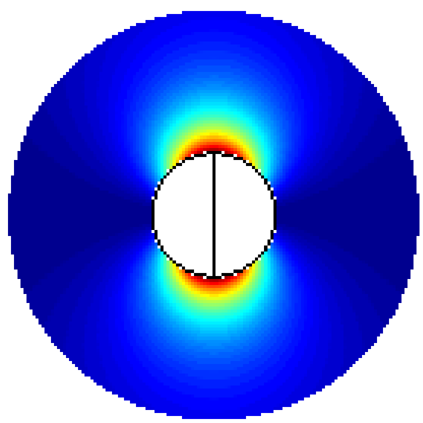
In addition, the presence of the $\cos^2\theta$ and $\sin^2\theta$ terms everywhere means that the stretching varies as you move north or south of the equator. Essentially the east-west stretching is maximum at the equator (where the most mass is spinning the fastest, and in the direction of the stretching) and smallest at the poles (where there is no spin). There is also north-south stretching that is largest at the poles (counter-intuitively, where there's little motion) and disappears at the equator (where the motion is fastest, but perpendicular to the direction of the stretch). But for an object like our sun, most of the stretching is still radial, just like for the non-spinning mass.

The following pictures demonstrate these results graphically, where the rotating mass is the white circle, and it is rotating *around the vertical line*. In other words, the equator of the circle (at "3 o'clock" and "9 o'clock") is rotating in and out of the page. Red is the most stretch, dark blue is the least. Colors (amount of stretch) are relative to each picture, so colors between pictures do not correspond to equal amounts of stretching. The θ or N/S stretch shows the amount of stretch <u>in the</u> (or) direction, while the φ or E/W stretch shows the amount of stretch in an arc perpendicular to the page (into or out of it, same as the direction of rotation).

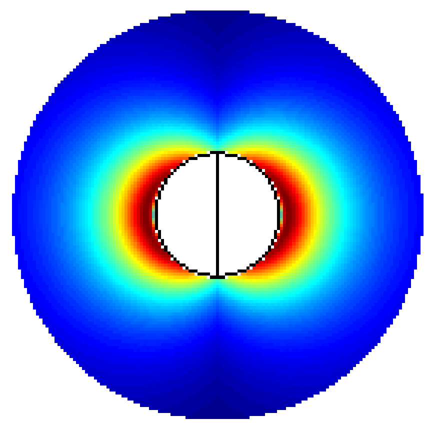
Radial stretch



Theta (N/S arc) stretch



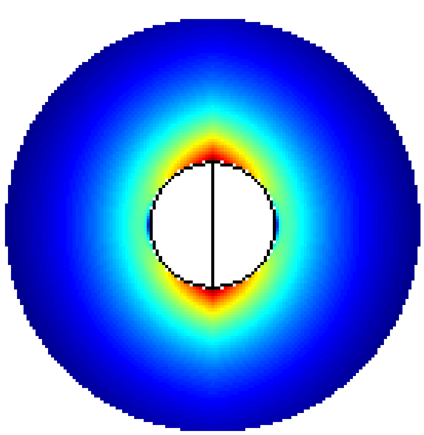
Phi (E/W arc) stretch



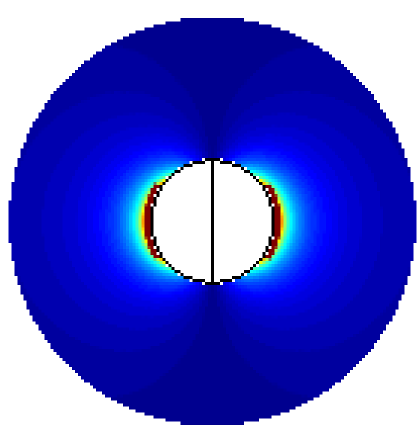
The amount that <u>time</u> is stretched is shown below, and it is greatest at the poles, where again counter-intuitively mass is moving slowest. But there is still a significant amount of stretching at the equator, as well.

The $dtd\varphi$ term in the metric means that a square in the "$t\varphi$ plane" would be stretched horizontally, vertically, *and diagonally*, producing a parallelogram. The last picture shows how much the <u>angle between</u> the time and φ (E/W) axes *differs from* 90 degrees. The angle between those axes gets smallest (light blue and yellow) near the equator (red indicates where the angle between the axes becomes *complex*, discussed more in GR2f).

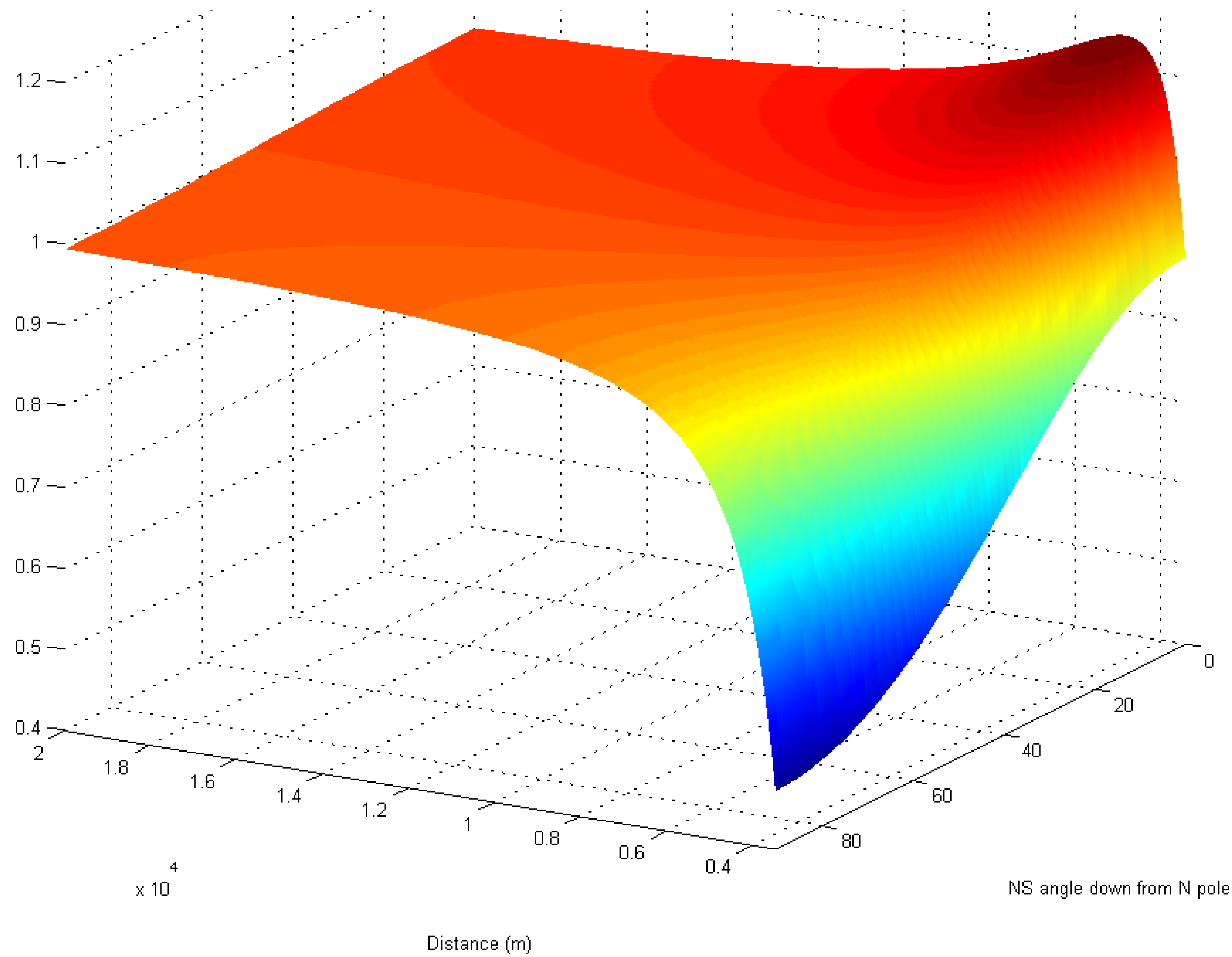
Time stretch



Time/phi angle



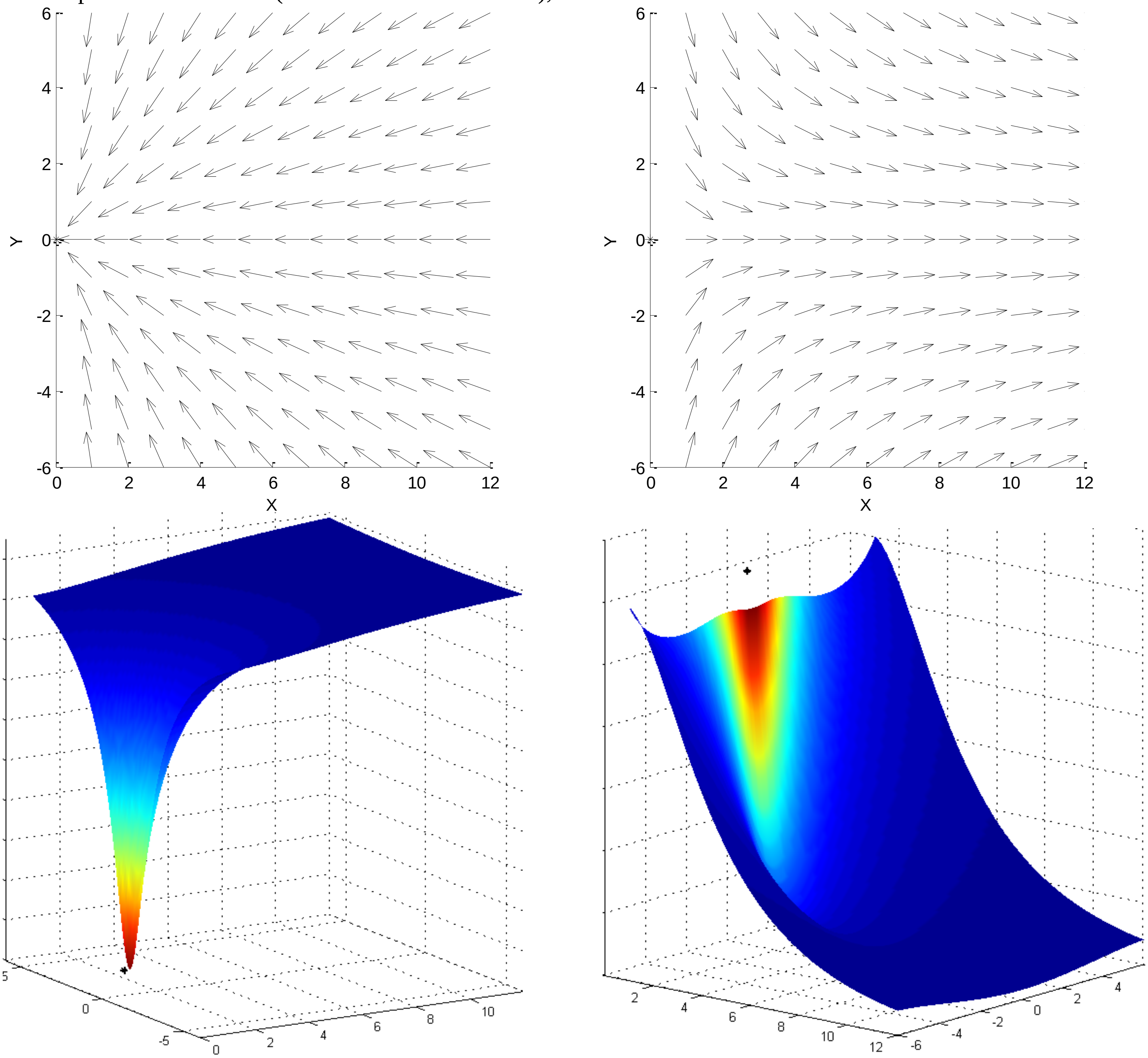
While we're talking about the Kerr spacetime, here's one more interesting thing about it : the amount of radial pull (straight down towards it) it has on objects it is slightly different than that of a non-rotating Schwarzschild mass. The following figure shows the ratio between the radial pull of a rotating and non-rotating object. Far from both objects the pull is the same (Z axis = ratio = 1), but closer in the rotating object has a slightly *larger* radial pull at the poles (the raised, darker red region), but a <u>much</u> *smaller* radial pull at the equator! The "radial" picture on the previous page shows this if you look carefully at the red region – it is slightly thicker at the poles than at the equator.



All these examples where the amount of stretching or pull is largest at the poles where very little mass is actually moving demonstrate the tensor nature of GR – mass moving in <u>one</u> direction can stretch space-time in <u>another</u> direction.

A Moving Mass

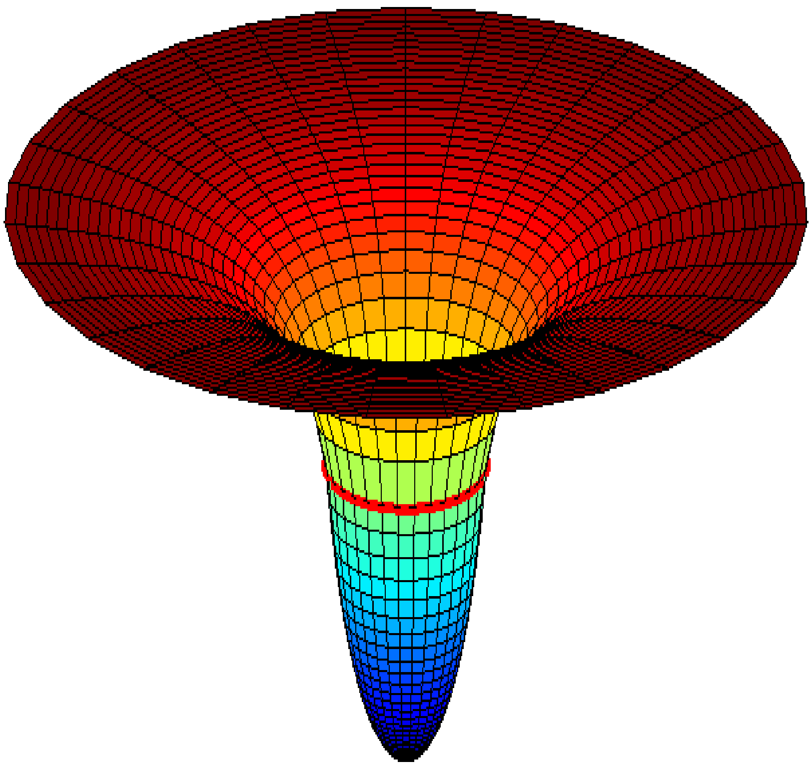
Another unusual result involving frame-dragging is that a fast-moving object actually repels other particles in front of it! This is in complete contradiction to Newtonian physics, which says that two objects can only be attracted to each other. It's as if spacetime "bunches up" in front of the object, creating a "hill" that forces particles to roll away from it. For a massive object at the origin (the *) slowly moving along the $+x$ axis, the left figure below shows the force that particles in front of it would feel, and they are all attracted to the object as expected (note : the arrows only show the direction of the force, not magnitude). But when the object is moving at 80% the speed of light, the right figure shows that all the particles are being forced away from the object in the X direction. The same results are shown in the figures below them, where the surfaces show which way a particle would "roll" (color is proportional to the amount of force). Note that these figures do not show the actual spacetime curvature (which is hard to visualize), but the *results* from the curvature.



The overall conclusion from both this and the Kerr result is that while a stationary mass creates a "dimple" in spacetime that draws objects towards it, a moving mass is "sticky" and grabs on to spacetime, stretching it in the direction of motion, and the stickiness (amount of stretch) is proportional to its speed.

Spacetime Inside a Non-Rotating Star

Einstein's equation has even been solved for a highly simplified model of the interior of a massive non-rotating object like a star or planet. Treating the object like a "perfect fluid" (which is very unrealistic), this picture shows that the amount of curvature drops to zero at the center of the object. When combined with the exterior Schwarzschild solution, we get a complete (approximate) picture of spacetime inside and outside the object. In the picture below, the curve of the surface (**not its depth**) represents the amount of curvature of space, so where the surface is horizontal at the very bottom of the well means space there is flat. The red circle in the green area is where the surface of the object is, so everything below that line is how spacetime curves inside the object.



The curvature is greatest (steepest) right at the surface of the object, but even for something as large as the earth, the spacetime **you're sitting in right now** is 99.99999993% perfectly flat! But even that little bit of curvature is enough to hold you firmly on the ground.

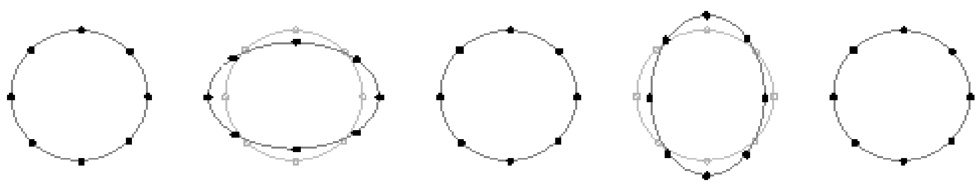
Other Solutions to Einstein's Equation

In addition to the Schwarzschild and Kerr solutions described above, there have been several other **exact solutions** found to Einstein's equation. These include an object with an electric charge on it, and a spinning object with a charge on it (which has been compared to the electron). There's also Schwarzschild and Kerr solutions with a non-zero Λ to incorporate the apparent acceleration of the expansion of the universe (but keep in mind that Λ is probably less than 10^{-50}, so its effects are very small except on cosmological scales). There are many other more obscure solutions that assume some symmetry, such as an infinitely long "line" of matter-energy, or rotating rings, disks, or spherical shells of matter-energy.

There are also exact solutions that model the **entire universe** as uniformly filled with matter that acts like "dust" and energy that acts like a perfect fluid (on cosmological scales, a particle of "dust" is an entire galaxy!). This model can then make predictions about the future of the universe depending on the density of matter in the universe, the ratio of matter to energy, whether spacetime is flat, positively or negatively curved, and the value of Λ, all of which can be inferred from astronomical observations. Current observations may indicate that the universe is very close to flat, and the expansion is accelerating (which means $\Lambda > 0$). This means that the universe must have started with a "big bang", and that it will continue to expand forever. See GR1f for more details.

Linearized gravity or the **weak-field approximation** assumes $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where η is the metric for flat unchanging Minkowski space and *h* is some small deviation (perturbation) from η. Since *h* (and its derivatives) are assumed to be small, all powers of *h* and its derivatives are ignored. This creates linear equations which are much easier to deal with, and effectively defines η as the background metric.

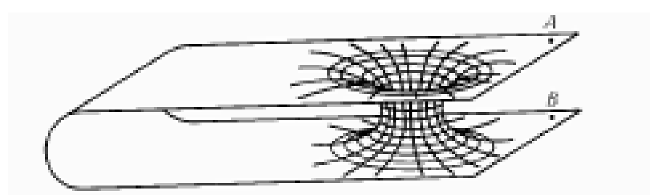
The weak-field approximation of Einstein's equation predicts the existence of **gravitational waves** that are given off by matter when it moves in certain (asymmetric) ways, such as exploding stars, merging pairs of neutron stars or black holes, or binary stars orbiting each other. The wave distorts matter as it passes by in the following manner (the wave is moving perpendicularly into the page) :



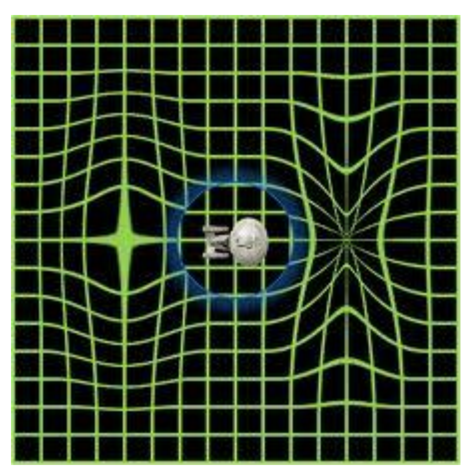
t_0 t_1 t_2 t_3 t_4 time →

It is also possible to design a specific shape of spacetime to get a certain result by first picking values of the metric tensor g_{uv} to get the desired spacetime curvature, then solving Einstein's equation for the stress-energy tensor $T_{\mu\upsilon}$ and finally finding a configuration of matter and energy that produces the required stress-energy tensor. Using this process, solutions to Einstein's equation have been found that represent "wormholes" and "warp drives".

A **wormhole** is basically two Schwarzschild black holes placed back-to-back. → To get to B, someone at point A would normally have to travel all the way to the left, then down and around, then all the way back to the right (remember, they must stay in the surface). Theoretically, travelling thru the wormhole would be a much shorter trip. But even if such a thing could be made (which is doubtful), and someone could travel thru it without being destroyed (which is unlikely), we have no idea how to control where the traveler would actually come out (or even if they would come back out in this universe)!



Warp drive metrics generally expand the space behind the source and contract the space in front of it → causing the source to move forward in such a way that someone outside the source would see it moving faster than the speed of light. In addition, the people inside the warp bubble would not feel any g-forces (so huge accelerations would be possible), and experience no time-dilation effects.

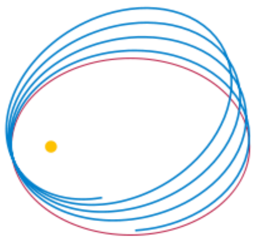


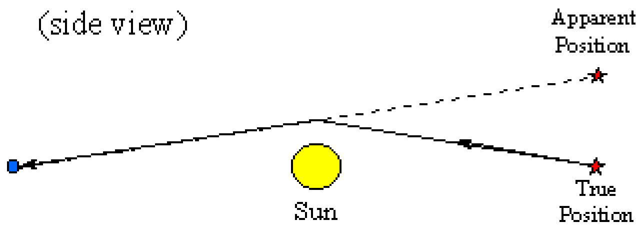
Altho technically they satisfy Einstein's equation, there are major problems with both wormholes and warp drives : they require kinds of matter that don't exist as far as we know, and/or impossibly large energies, in order to create and maintain them. The kind of matter needed would have to be *repelled* by a gravitational field (called "negative mass", which is not antimatter). For the energy requirements, one calculation concluded that converting all the matter in the *observable universe* into energy would not be enough to create and sustain a warp bubble! There's also some question as to whether or not wormholes and warp drives violate various energy conservation requirements (discussed in GR2f).

Predictions and Tests

General relativity makes many predictions, which have been tested many times in many ways over the years, and it has always been successful :

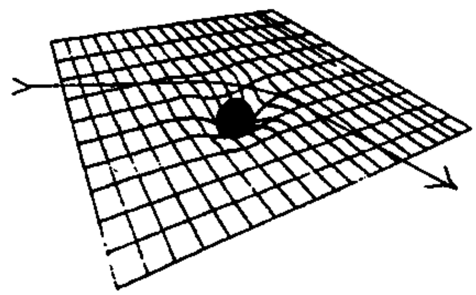
- Mercury's orbit is an ellipse, and its orientation changes position over time ("perihelion precession"). → When all other Newtonian effects were accounted for, there was still a little bit of precession left over. The amount predicted by relativity accounted for that perfectly. This has been confirmed with binary pulsars as well. Note – this effect is not due to frame-dragging.



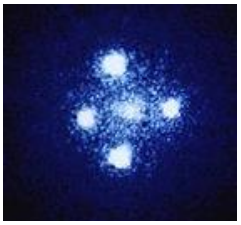


- The prediction that a massive object would bend light around it was tested by observing the positions of the stars in the sky, then observing their positions during a solar eclipse. → Their apparent positions moved exactly as much as general relativity said they would.

  The reason that light bends is that it is actually following a geodesic path thru the warped space around the sun. →

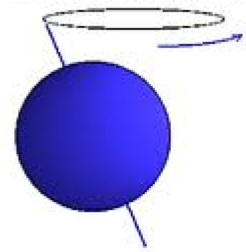


  "Gravitational lensing" is an extreme form of this, where one object gets split into multiple images (the four north/south/east/west dots, which are all the same galaxy) by another object in front of it (the center dot, which is another galaxy).



- Light leaving a massive object (like a star) will lose energy as it "climbs uphill" away from the source, and so its frequency will decrease ("gravitational redshift"). This has been confirmed both in laboratory experiments and with observations of a variety of stars.
- The prediction that light takes longer to travel thru a gravitational field (because the space is stretched) is called the "Shapiro delay", and has been tested and confirmed within our solar system by bouncing radar signals off of Mercury and Venus, and from signals sent to Earth by various Mars probes.
- As shown in both the Schwarzschild and Kerr solutions above, time slows down in a gravity field ("gravitational time dilation"), and the effect increases the further down into the gravity well you go. This was originally confirmed by flying atomic clocks in airplanes at high altitudes and comparing their times to other clocks that stayed on the surface of the Earth.
- In an expanding universe, general relativity predicts that light between two *comoving* objects also gets redshifted, from the expansion itself ("cosmological redshift"). This effect was first observed in 1929, and has nothing to do with whether the objects are moving towards or away from each other in space.

- Despite many very sensitive tests over many years, gravitational waves have not yet been detected by laboratory equipment. But they have been *indirectly* observed in a binary pulsar – the rate of slowing in the stars' orbital period is exactly equal to the predicted energy loss if they were radiating gravitational waves.
- The prediction of a geodetic (or "de Sitter") precession of the spin of an orbiting object → is due to the effects of parallel transport in a curved spacetime, and does not show up at all in Newtonian gravity (it is also not due to frame-grabbing). This was tested and verified just recently by a satellite called "Gravity Probe B" orbiting the Earth.



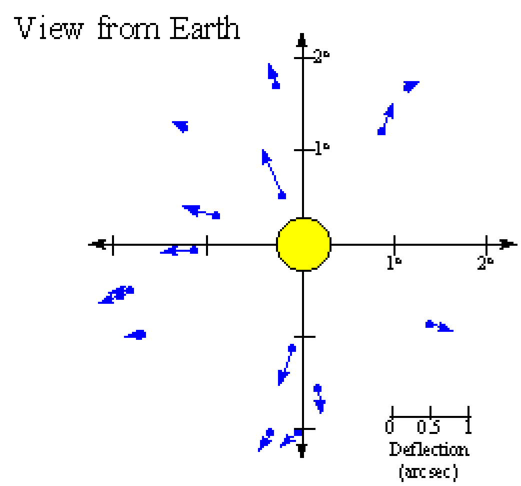
- Frame-dragging has probably been confirmed (there is some debate) thru extensive precision measurements of the moon's orbit. Gravity Probe B was also supposed to test precession due to frame-dragging, but was initially unable to conclusively prove it due to unexpected technical difficulties. However, after years of data-processing, the researchers now claim that it has detected frame-dragging.

There have been many minor (and major) suggested modifications to general relativity over the years, as well as completely different theories of gravitation. None of them match the observed data as well as general relativity. But it bothers some physicists that there are solutions to Einstein's equation that are not physically possible (like wormholes and warp drives), or solutions whose physical meaning cannot yet be understood (like an infinite line of matter). And the fact that general relativity and quantum mechanics cannot be combined is really annoying….

Applications

The accuracy of the atomic clocks in the satellites in orbit around the Earth for the GPS system is so high that they require a correction due to the fact that their clocks are running at a different speed in orbit than clocks on the ground, due to gravitational time dilation.

Astronomical measurements of stars' positions are getting so accurate that the distortion in their observed position by the bending of spacetime by the sun has to be corrected, even when the stars aren't near the sun! →



Gravitomagnetism

"Gravitomagnetism", the "Lense-Thirring effect", or sometimes "gravimagnetism" are alternate names used to describe the frame-dragging effect caused by rotating objects. Since frame-dragging creates a force on a moving particle that is perpendicular to the particle's motion, much in the same way that a magnetic field creates a force on a moving charged particle, the gravitational force due to frame-dragging was dubbed "gravitomagnetism".

Using the weak-field approximation, Einstein's equation can be simplified to a form very similar to Maxwell's equations for electromagnetism (sometimes called "gravitoelectromagnetism"). In this format, the "electric" component is the "regular" gravity due to an object's mass, and the "magnetic" component is the gravity due to frame-dragging when the object moves (for example, spins). For comparison, the Earth's gravitomagnetic field at the equator is only about one-millionth as strong as its "regular" or gravitoelectric gravity field.

This simplified version of Einstein's equation also indicates that a *changing* gravitoelectric field creates a gravitomagnetic field, and a *changing* gravitomagnetic field creates a gravitoelectric field, leading to gravitational waves (just like a changing electric field creates a magnetic field, and a changing magnetic field creates an electric field, leading to electromagnetic waves).

Gravity and Electromagnetism

General relativity predicts that in magnetized plasmas, gravitational and electromagnetic waves may exchange energy between themselves and with the plasma. In addition, when an electromagnetic wave passes by a charged black hole, the electromagnetic wave can be converted to a gravitational wave. This means that electrically or magnetically charged moving matter can convert electromagnetic energy into gravitational energy, and vice-versa.

General relativity also predicts that electric or magnetic fields (which are, after all, energy) warp spacetime a very tiny bit, but the amount of stretching is undetectable at the field strengths we can produce in a lab. Likewise, electric currents (which also involve moving mass) create a very tiny gravitomagnetic field.

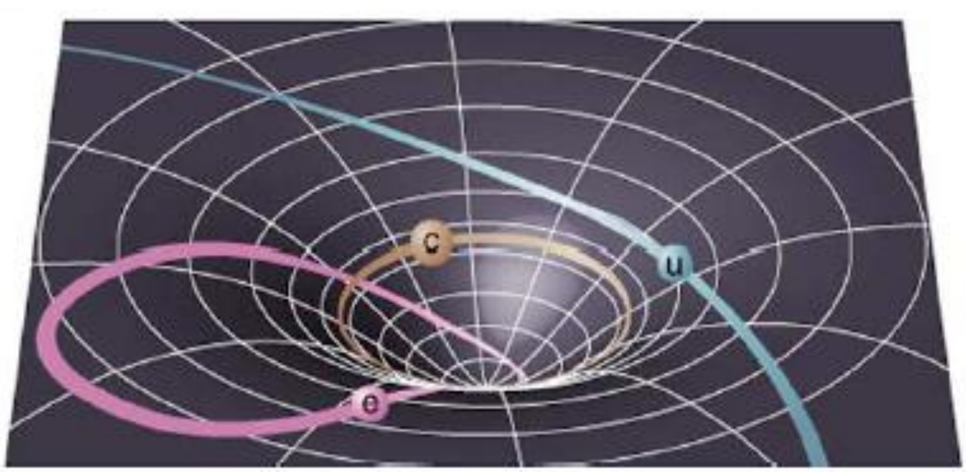
A controversial experiment (but which has been published in a peer-reviewed journal) claims to have detected a gravity-shielding (!) effect using a rotating disk carrying a superconducting current in a changing magnetic field. So it is possible that electricity, magnetism, and moving matter interact in ways not yet fully understood.

An even more tentative effect, called "gravitational magnetism" or the "Blackett effect", speculates that rotating masses create a magnetic field. If true, this effect does not appear to be explained by general relativity. But it has been claimed that this effect appears when torsion is included in the gravitational equation.

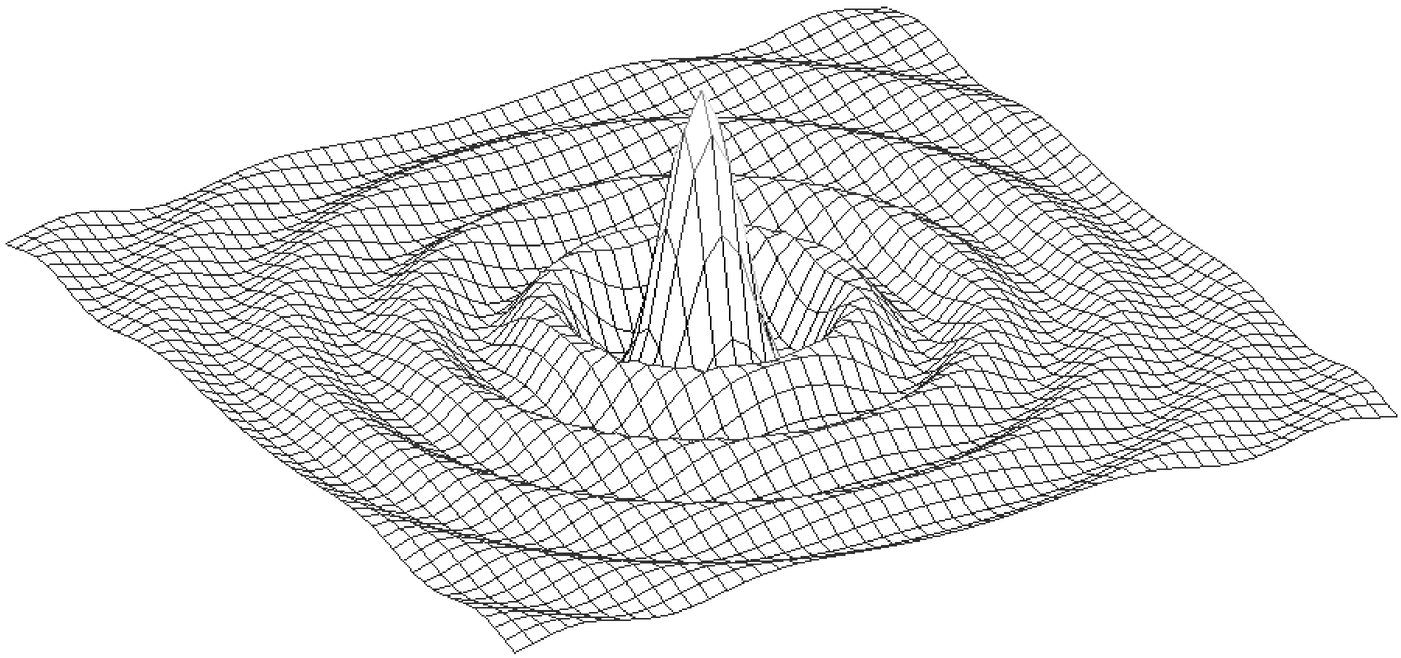
Other "Examples" of Curved Spacetime

Unlike the earlier images, these are not exact solutions as calculated by general relativity, but at least they give some idea of the ways in which spacetime can be curved.

Elliptical (pink), Circular (orange), and Unbound (blue) orbits in the curved spacetime around a planet or star :



Gravitational waves (literally ripples in spacetime), as produced by a supernova, binary star system, or collapsing star :



Keep in mind that the 2-D creatures living in this surface would not know that their space is being stretched, because they are stretching with it!